PARTIAL ANALYTICITY AND NODAL SETS FOR NONLINEAR ELLIPTIC SYSTEMS

HERBERT KOCH AND NIKOLAI NADIRASHVILI

Abstract. We study partial analyticity of solutions to elliptic systems and analyticity of level sets of solutions to nonlinear elliptic systems. We consider several applications, including analyticity of flow lines for bounded stationary solutions to the 2-d Euler equation, and analyticity of water waves with and without surface tension.

1. Introduction

For nonlinear elliptic equations and elliptic systems given by functions depending analytically on their arguments any sufficiently regular solution defined in an open domain is necessarily a real-analytic function. This classical result was established in the pioneering works of S. Bernstein [3], H. Levy [18], I. Petrovsky [23] and Morrey [22]. Our goal is to generalize these results to nonlinear elliptic systems which are analytic with respect to a group of spatial variables: we assume that the dependence of the equations on spatial variables is real-analytic only for a part of variables. Under such an assumption we prove that the solutions are analytic in the same group of variables. We call that property the partial analyticity.

We are interested in analytic solutions to nonlinear elliptic systems

$$\sum_j \partial_j F_k^j(x, u_i, Du) = f_k(x, u, Du) \qquad 1 \le k \le N, \tag{1}$$

where $u : D \to \mathbb{R}^N$, D is a bounded domain in $\mathbb{R}^n$, F_k and f_k are real. We assume that F_k^j and f_k are defined on an open set

$$U \subset \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{n \times N}$$

and we define

$$a_{kl}^{ij}(x, u, p) = \partial_{p_j^l} F_k^i(x, u, p)$$

where (x, u, p) denotes a typical element of the domain of definition, which may sometimes be ambiguous since we also use u as a notation for a function $u : D \to \mathbb{R}^N$.

Definition 1.1. *We call the coefficients* $(a_{kl}^{ij}) \in \mathbb{R}^{n\times n\times N\times N}$ *elliptic if for all* $0 \neq \xi \in \mathbb{R}^n$ *the matrix*

$$A_{kl}(\xi) = \sum_{i,j=1}^n a_{kl}^{ij} \xi_i \xi_j$$

is invertible. We call (1) *elliptic if there exists* C *so that*

1991 *Mathematics Subject Classification.* 35J47.

The authors acknowledge support by the DFG through SFB 1060 and the Alexander von Humboldt foundation for the second author.

(1) The coefficients are bounded: $|a^{ij}_{kl}(x,u,p)| \le C$.
(2) The inverses are bounded: $\|(A_{kl}(x,u,\xi))^{-1}\| \le C|\xi|^{-2}$.

Let $n_1+n_2=n$, $1 \le n_1, n_2 < n$ and denote $x=(x',x'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. The main object of interest is analyticity with respect to x'. There are different equivalent definitions of analyticity: Bounds on the growth of derivatives, locally uniform convergence of power series, and extension to a holomorphic function on a complex domain. All this is qualitative, whereas proofs often rely on quantitative formulations of analyticity.

With this notation we formulate the first main result where we consider a small perturbation of a constant coefficient system. This can be achieved by localizing and rescaling for partially analytic elliptic systems with C^1 regularity. Trivially every affine function $u:\mathbb{R}^n \to \mathbb{R}^N$ satisfies

$$\sum_{i,j,l} \partial_i \left(a^{ij}_{kl} \partial_j u^l \right) = 0$$

for all k. We consider solutions as perturbations of affine functions.

Theorem 1. *Let* $0<s<1$, (a^{ij}_{kl}) *be elliptic,* $\bar u \in \mathbb{R}^N$, $b \in \mathbb{R}^{n\times N}$. *There exist* $\delta>0$, $\rho>0$ *and* ε *depending only on* n *and* a^{ij}_{kl} *such that the following is true. If*

(i) Let

$$V \subset (\mathbb{C}^{n_1} \times \mathbb{R}^{n_2}) \times \mathbb{C}^N \times \mathbb{C}^{n\times N}$$

contain a 2ε *neighborhood of*

$$\left\{ \left(x'+iy', x'', \bar u + b \begin{pmatrix} x'+iy' \\ x'' \end{pmatrix}, b \right), x=(x',x'') \in \overline{B_1(0)}, |y'| \le \delta(1-|x|^2)^3_+ \right\}.$$

We assume that $F^j_k, f_k : V \to \mathbb{C}$ *are bounded measurable functions, holomorphic in* $z'=x'+iy'$, u *and* p *for almost all* x''.

(ii) The inequality

$$|F^i_k(z',x'',u,p) - F^i_k(\tilde z', \tilde x'', \tilde u, \tilde p) - a^{ij}_{kl}(p-\tilde p)| \le \rho(|x''-\tilde x''|^s + |u-\tilde u| + |z-\tilde z|) + \varepsilon|p-\tilde p|$$

holds in V. *Moreover*

$$\|f_k(z',x'',u,p)\|_{sup} + \|D_{u,p} f_k\|_{sup} < \rho.$$

(iii) The weak solution $u \in C^{0,1}(B_2(0),\mathbb{R}^N)$ *to* (1) *satisfies*

$$\|u(x) - (\bar u + b\cdot x)\|_{C^{0,1}(B_2(0))} \le \varepsilon.$$

Then the solution u *has a unique extension* $u^{\mathbb{C}}$ *to*

$$B_1^{\mathbb{C}} := \left\{ (x'+iy', x'') : |(x',x'')| \le 1, |y| \le \delta(1-|x|^2)^3_+ \right\}$$

such that

$$\|u^{\mathbb{C}} - (\bar u + b(x+iy'))\|_{\sup, B_1^{\mathbb{C}}} + \|Du^{\mathbb{C}} - b\|_{\sup, B_1^{\mathbb{C}}} < 2\varepsilon$$

and u *is holomorphic with respect to* $x'+iy'$ *for all* x''.

Remark 1.2. *By the Cauchy integral this implies estimates*

$$|\partial^\alpha D_x u(x)| \le c|\alpha|! R^{-|\alpha|} \tag{2}$$

for every multiindex α *corresponding to* $\mathbb{R}^{n_1}$

$$R = \delta(1-|x|^2)^3_+/2.$$

We call u uniformly partially analytic if c and R can be chosen uniformly on compact sets.

The smallness condition can be achieved if u, F and f are continuously differentiable at any point x by choosing a small ball around x and rescaling it to size 1. This will be done in detail in the proof of Theorem 3. There is not much we know about the size of δ even in simple situations: Consider the homogeneous Laplace equation. Then we may choose any $\delta < 1$, but we do not know whether this is optimal, nor whether more is known.

If $n_2 = N = 1$ much weaker smallness conditions are sufficient for a similar conclusion.

Theorem 2. *Let $n \geq 2$, $(a^{ij}(x_n))$ be a measurable one parameter family of bounded and uniformly elliptic coefficients, $b \in \mathbb{R}^{n-1}$, $\bar{u} \in C^{0,1}((-2,2))$ and $J \subset I \subset \mathbb{R}$ intervals, where J is compact and I is open. Then there exist $\delta > 0$, $\varepsilon > 0$ and $\rho > 0$ depending only on n, the bound and the ellipticity constant of the coefficients such that the following is true.*

(i) Suppose that the open set

$$V \subset (\mathbb{C}^{n_1} \times \mathbb{R}) \times \mathbb{C} \times \mathbb{C}^n$$

contains a 2ε neighborhood of (with I corresponding to $\partial_n u$)

$$\left\{(x' + iy', x_n, \bar{u}(x_n) + b \cdot (x' + iy'), b) : |x| \leq 1, |y'| \leq \delta(1 - |x|^2)_+^3\right\} \times I.$$

We assume that $F^j, f : V \to \mathbb{C}$ are bounded measurable functions, holomorphic in x', u and p for almost all x_n.

(ii) The inequality

$$|F^i(z', x_n, u, p) - F^i(z', x_n, \widetilde{u}, \widetilde{p}) - a^{ij}(x_n)(p_j - \widetilde{p}_j)| \leq \rho|u - \widetilde{u}| + \varepsilon|p - \widetilde{p}|$$

holds for points in V and

$$|f| + |D_{u,p} f| \leq \rho$$

(iii) Let $u \in C^{0,1}(B_2(0))$ satisfy

$$\left\| u(x) - (\bar{u}(x_n) + \sum_{j=1}^{n-1} b_j x_j) \right\|_{C^{0,1}(B_1)} \leq \varepsilon$$

$$\frac{\bar{u}(x_n) - \bar{u}(\widetilde{x}_n)}{x_n - \widetilde{x}_n} \in J$$

and, in a distributional sense,

$$\partial_i F^i(x, u, Du) = f(x, u, Du).$$

Then u has a unique extension $u^{\mathbb{C}}$ to $B_1^{\mathbb{C}}$ such that

$$\|u^{\mathbb{C}} - (\bar{u}(x_n) + \sum_{j=1}^{n-1} b_j(x_j + iy_j))\Big\|_{sup, B_1^{\mathbb{C}}} + \|Du^{\mathbb{C}} - (b, \partial_n \bar{u})\Big\|_{sup, B_1^{\mathbb{C}}} < 2\varepsilon$$

and u is holomorphic with respect to $x' + iy'$ for almost all x''.

Remark 1.3. *The analogous result holds for Dirichlet boundary conditions and for elliptic boundary value problems for a boundary given by $x_n = 0$. The proof covers diagonal systems and general coercive equations if $n = 2$. Note however the following example. The function*

$$u(x_1, x_2) = \arctan(x_1/x_2)$$

is harmonic in $x_2 > 0$, it satisfies

$$u(x_1, 0) = \begin{cases} \pi/2 & \text{if } x_1 > 0 \\ -\pi/2 & \text{if } x_1 < 0 \end{cases} =: h(x_1).$$

The derivative

$$\partial_{x_2} u(x_1, x_2) = \frac{x_1}{x_1^2 + x_2^2}$$

is unbounded. In particular it is not uniformly analytic in x_2 near $x_2 = x_1 = 0$ for fixed x_1. A small modification gives a counterexample for homogeneous Dirichlet data. Let

$$v(x_1, x_2) = \int_0^{x_2} \arctan(x_1/t)dt$$

It satisfies $v(x_1, 0) = 0$ and

$$\begin{aligned} \Delta v &= \int_0^{x_2} \partial_{x_1}^2 \arctan(x_1/t)dt + \partial_{x_2} \arctan(x_1/x_2) \\ &= \int_0^{x_2} \Delta \arctan(x_1/t)dt + h(x_1), \end{aligned}$$

hence

$$\Delta v = h(x_1), \qquad v(x_1, 0) = 0.$$

Clearly $\partial_{x_2}^2 v$ is not uniformly bounded and hence v is not uniformly analytic in x_2 near $x_1 = x_2 = 0$. Notice that in the classical case of complete analyticity of the equation solutions are analytic up to the boundary [22]. The example shows that for partial analyticity it is important that the boundary is tangential to the analytic direction.

The main tool of the proof of Theorem 1 and Theorem 2 is a complexification of the equation in a group of variables in which the analyticity holds. The classical approach of Levy, Petrovsky and Morrey is also based on the complexification of the equations, however their methods essentially used the whole set of variables. Our argument applies to any number of variables and it gives an independent proof of analyticity for analytic elliptic boundary value problems.

Corollary 1.4. *Suppose that F and f are analytic in all arguments, $u \in C^1$ satisfies the elliptic equation*

$$\partial_i F^i(x, u, Du) = f(x, u, Du).$$

Then u is analytic. The same result holds for analytic elliptic boundary value problems.

Proof. The first statement follows from Theorem 1 with $n_2 = 0$, possibly after localizing to a small neighborhood of a point and rescaling. We did require $n_2 > 0$, but there is no change of the proof for $n_2 = 0$. The argument for analyticity in tangential directions for boundary value problems is the same as for partial

analyticity, with the difference that the notion of ellipticity is harder to formulate and we refer to [1] and [22] for a discussion.

The theorem of Cauchy-Kowalevskaja [16] implies existence of an analytic solution for analytic Cauchy data. In particular it gives the correct bounds of any derivative in terms of the analytic Cauchy data. We use these bounds at each level x_n assuming that the boundary is given by $x_n = 0$. $\square$

Theorems 1 and 2 have important consequences for level sets of solutions u. Roughly speaking the regularity statement does not change if we do a coordinate change in dependent and independent variables, which interchanges the role of partial analyticity and analyticity of level sets.

Theorem 3. *Let $n = n_1 + n_2$, $N = n_2$, $0 < s < 1$, $U \subset \mathbb{R}^n$ be open and let $u \in C^1(U, \mathbb{R}^N)$ be a weak solution to the equation*

$$\partial_i F^i_k(x, u, Du) = f_k(x, u, Du).$$

We assume that

(i) The open set $V \subset \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{n\times N}$ contains the closure of

$$\{(x, u(x), Du(x)) : x \in U\}.$$

The functions $F^i_k(x, u, p)$ are continuous and the functions $f_k(x, u, p)$ are measurable and uniformly analytic with respect to x and p, more precisely there exists R and c such that

$$\sup_\alpha R^{|\alpha|} \left\|\partial^\alpha_{x,p} F^i_k\right\|_{C^s} \le c \qquad \text{and} \sup_{x,u,p,\alpha} R^{|\alpha|} \sup_{x,p} |\partial^\alpha_{x,p} f_k(x, ., p)| \le c.$$

(ii) The coefficients

$$a^{ij}_{kl}(x) = \frac{\partial F^i_k}{\partial p^j_l}(x, u(x), Du(x))$$

are elliptic.

(iii) Du has rank N in U.

Then the level sets

$$\{x \in U | u(x) = y\}$$

are uniformly analytic for all y on compact sets. Moreover Du is uniformly analytic when restricted to a compact subset of a level set: i.e. if $W \subset \mathbb{R}^{n_1}$ open, $\phi : W \to \mathbb{R}^{n_2}$ is an analytic function whose graph is in U such that $x' \to u(x', \phi(x'))$ is a constant function then

$$x' \to Du(x', \phi(x'))$$

is analytic uniformly in the level.

The continuity assumption can be relaxed if $N = 1$.

Theorem 4. *Let $u \in C^{0,1}(U)$ be a weak solution to*

$$\partial_i F^i(x, u, Du) = f(x, u, Du).$$

Suppose that

(i) The set $V \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ contains the closure of

$$\{(x, u(x), Du(x)) : x \in U\}$$

and, if $(x,u,p) \in V$ and $(x,u,q) \in V$ then also $(x,u,p+t(q-p)) \in V$ for $0 \le t \le 1$. The functions F and f are uniformly analytic in V with respect to x and p in the sense that there exists $C, R > 0$ such that

$$R^{|\alpha|}\left(\|\partial^\alpha_{x,p} F^i(x,u,p)\|_{sup} + \||\partial^\alpha_{x,p} f|\|_{sup}\right) \le c.$$

(ii) The coefficients

$$a^{ij}(x) = \frac{\partial F^i}{\partial p^j}(x,u(x),Du(x))$$

are uniformly positive definite.

(iii) u is uniformly monotone in the direction $0 \ne v \in \mathbb{R}^n$, i.e.

$$\inf_{x,h} \frac{u(x+hv)-u(x)}{h} > 0.$$

Then all level sets of u in U are uniformly analytic. Moreover for almost all levels u is differentiable at this level set, and the restriction of Du to the level set is uniformly analytic on compact sets.

In Section 2 we consider applications of Theorem 3 to problems from different areas of analysis. Even though the theorems are quite general and flexible we consider the techniques to be applicable in many more situations. This is illustrated in some of the examples where we rely on modifications of the proofs instead of an application of the theorems.

The proof of the Theorems follows in the remaining half of the paper: Singular integral estimates in Section 3 state standard results and a crucial variant needed for Theorem 2 and Theorem 4. Partial analyticity for solutions to linear elliptic equations to the theme of Section 4. The corresponding result for nonlinear equations follows by a standard fixed point argument in Section 5, where Theorem 1 and Theorem 2 are proven. The paper is completed with Section 6 where Theorem 3 and Theorem 4 are deduced from Theorem 1 and Theorem 2 via a change of coordinates - with the important feature that it involves dependent and independent coordiantes.

The theorems of this introduction are the main abstract results and we number them by single digits. The remaining numbering is done counting within the section. We use the summation convention, but without a consistent use of upper and lower indices.

2. Examples

In this section we discuss several examples arising from different fields of analysis and physics.

2.1. **Lagrangian trajectories of fluid dynamics.** Smoothness of streamlines (trajectories of material particles of the fluid) is a classical subject. For many hydrodynamical models solutions in Eulerian variables (velocity, pressure) are less regular than in Lagrangian variables.

The study of regularity of streamlines goes back to the work of Lichtenstein [19], Chemin and Serfati. For historical surveys of the problem see Majda and Bertozzi [21]. With a maximal regularity approach the regularity of streamlines was settled in Serfati [24] and again taken up in Zheligovsky and Frisch [26] and Constantin, Vicol and Wu [8]. The authors proved that for the Euler equation of ideal fluid, for the quasi-geostrophic equation and for the Boussinesq equation that if the classical

solutions are defined on a time interval and have Eulerian velocities in $C^{1,\varepsilon}$, $\varepsilon > 0$ then their Lagrangian paths are real analytic.

On the opposite extreme, existence of flow lines has been proven by Bianchini and Gusev [4] under very weak assumptions on the velocity vector field.

We discuss in more detail the Euler equation of an ideal fluid in dimension 2 where we will obtain a related result under considerably weaker regularity assumptions. Let

$$v(t,x) = (v_1(t,x), v_2(t,x)), \quad t \in \mathbb{R}, x \in \mathbb{R}^2$$

be the velocity and $p(t,x)$ the pressure of a solution of the Euler equation for an ideal fluid:

$$\begin{cases} \partial v/\partial t + v\nabla v = -\nabla p, & \text{in } \mathbb{R}^2 \times \mathbb{R} \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^2 \times \mathbb{R} \end{cases} \tag{3}$$

For any initial data $v_0 \in C_0^{k,s}(\mathbb{R}^2)$, $k = 1, 2, ..., 0 < s < 1$

$$v(x,0) = v_0(x)$$

there exists a unique solution $v(t,x)$ of (1) defined for all $t \in \mathbb{R}$ such that for all $t \in \mathbb{R}$ $v \in C^{k,s}(\Omega)$, see [20, 21, 14].

The vector field $v(x,t)$ defines a flow $z_x(t)$ on $\mathbb{R}^2$, which is a one-parametric group of area preserving diffeomorphisms of $\mathbb{R}^2$. Let $x_0 \in \mathbb{R}^2$. The curve $z_{x_0}(t) \in \mathbb{R}^2$,

$$\gamma : t \in \mathbb{R} \to z_{x_0}(t) \in \mathbb{R}^2$$

is called the streamline of a material particle with initial position x_0 of the fluid.

From the results of Serfati [24] it follows that if the initial velocity v_0 is in $C^{1,\varepsilon}$, $\varepsilon > 0$ then for any z_0 the map γ is real analytic. The assumption that $v \in C^{1,\varepsilon}$, $\varepsilon > 0$, is essential. The following is an instance of Theorem 3 where the assumptions on the regularity of v are considerably weakened for stationary flows.

Theorem 2.1. *Suppose that v is a bounded stationary solution to the 2-d Euler equation* (3) *on the unit disc D and that there exists $\delta > 0$ with $v_1 > \delta$. Then there exists $\mu = \mu(\|v\|_{L^\infty}/\delta)$ and a Lipschitz map*

$$\psi : \left\{(x_1 + iy_1, x_2) \in \mathbb{C} \times \mathbb{R} : |x_1|^2 + |y_1/\mu|^2 + x_2^2 \leq \frac{1}{4}\right\} \to \mathbb{C}.$$

It is uniformly holomorphic in z_1 for (z_1, x_2) in compact sets, $\psi(0, x_2) = x_2$ and for almost all x_2 and all x_1

$$v_1(x_1, \psi(x_1, x_2))\psi_{x_1}(x_1, x_2) - v_2(x_1, \psi(x_1, x_2)) = 0. \tag{4}$$

Moreover for almost all x_2 the map

$$x_1 \to v(x_1, \psi(x_1, x_2))$$

is analytic, again uniformly in compact sets (for x_1 and x_2).

The identity (4) implies that the vector field is tangential to the graph of $x_1 \to \psi(x_1, x_2)$ and that there is an analytic flow line. The simplest such flows are shear flows $v = (v_1(x_2), 0)$, with v_1 bounded from below and above. The flow lines are horizontal curves, and the restriction of the velocity to a flow line is defined for almost all x_2, and it is constant and hence analytic whenever it is defined.

This theorem is vague about what we require from v to call it a solution to the stationary Euler equation, and this is an important point we want to clarify. The simplest definition would be a distributional solution to

$$\left\{ \begin{array}{ll} \sum_{i=1}^2 \partial_i(v_i v_j) = -\partial_j p, & \text{in } \mathbb{R}^2 \times \mathbb{R} \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^2 \times \mathbb{R} \end{array} \right. \tag{5}$$

but for technical reasons we are not able to work with this definition. There is a formulation which may be more relevant for the physical interpretation. Let

$$e = p + \frac{1}{2}|v|^2$$

be the inner energy. Bernoulli's law states that it is constant along flow lines. We calculate using Euler's equation

$$\begin{aligned} \nabla e =& \nabla p + \begin{pmatrix} v_1 \partial_1 v_1 + v_2 \partial_1 v_2 \\ v_1 \partial_2 v_1 + v_2 \partial_1 v_1 \end{pmatrix} \\ =& (\partial_1 v_2 - \partial_2 v_1) \begin{pmatrix} v_2 \\ -v_1 \end{pmatrix}. \end{aligned} \tag{6}$$

The last vector is perpendicular to the velocity vector field and this implies Bernoulli's law. The last identity can be rewritten in divergence form

$$\begin{aligned} \partial_1(\frac{1}{2}(v_1^2 - v_2^2)) + \partial_2(v_1 v_2) + \partial_1 e =& 0 \\ \partial_2(\frac{1}{2}(v_2^2 - v_1^2)) + \partial_1(v_1 v_2) + \partial_2 e =& 0 \\ \nabla \cdot v =& 0 \end{aligned} \tag{7}$$

Since the velocity vector field is divergence free there exists a velocity potential Φ with $v = \begin{pmatrix} \partial_2 \Phi \\ -\partial_1 \Phi \end{pmatrix}$, since $\nabla \cdot v = 0$, which is Lipschitz continuous if the velocity vector field is bounded. Then the system reads as

$$\begin{aligned} \frac{1}{2}\partial_1(\Phi_2^2 - \Phi_1^2) - \partial_2(\Phi_1 \Phi_2) + \partial_1 e =& 0 \\ \frac{1}{2}\partial_2(\Phi_1^2 - \Phi_2^1) - \partial_1(\Phi_1 \Phi_2) + \partial_2 e =& 0 \end{aligned} \tag{8}$$

combined with

$$\Delta e = \frac{1}{2}\left(\partial_{11}^2(\Phi_1^2 - \Phi_2^2) + \partial_{22}^2(\Phi_2^2 - \Phi_1^2)\right) + \partial_{12}^2(\Phi_1 \Phi_2).$$

This formulation is related to the one used by Delort [9] and Evans and Müller [10]. Up to this point we did only use that $\Phi \in H^1_{loc}$. Now suppose that $u \in L^\infty \cap W^{1,1}$ has integrable derivatives. Then by (6) e has integrable derivatives and ∇e and $\nabla \Phi$ are linearly dependent. If in addition $\partial_2 \Phi > \delta > 0$ then e is a function of Φ, $e = e(\Phi)$ for some function e and Φ satisfies the elliptic equation

$$\partial_2(\frac{1}{2}(\Phi_1^2 - \Phi_2^2)) - \partial_1(\Phi_1 \Phi_2) + \partial_2 e(\Phi) = 0 \tag{9}$$

which, under the assumption that Φ has integrable second derivatives becomes

$$\Delta \Phi = e'(\Phi)$$

for an integrable function e'. By a weak solution in Theorem 2.1 we mean a weak solution to (9) for some local integrable function $\Phi \to e$. If (v, e) is a weak solution

to (7), if Φ is the velocity potential and if $v_1 > \kappa > 0$ then Bernoulli's law may be formulated as the requirement that e is a function of Φ which implies (9).

The claim of the Theorem 2.1 is a consequence of Theorem 4 without assuming anything on e beyond local integrability. Only boundedness of the inner energy e has to be deduced from the upper and lower bounds on the velocity.

Suppose that $\Phi \in C^{0,1}$ and e be functions which satisfy (9) and $\Phi_2 > \delta$. Then we obtain a bound on the oscillation of e in terms of δ and the Lipschitz constant. We choose a non-negative test function $h \in C_0^\infty(-1/2, 1/2)$ and integrate $h(x_1)$ over $t_1 \le \Phi \le t_2$. We apply the divergence theorem to (9) multiplied by h:

$$\frac{1}{2}\int_{\Phi=t} \frac{1}{2}|D\Phi|\Phi_2 h(x_1) d\mathcal{H}^1 \Big|_{t=t_1}^{t=t_2} - \int_{t_1<\Phi<t_2} \Phi_1\Phi_2 \partial_{x_1} h(x_1) dx$$
$$= \int_{\Phi=t} h(x_1) \frac{\Phi_2}{\sqrt{\Phi_1^2+\Phi_2^2}} d\mathcal{H}^1 e(t) \Big|_{t=t_1}^{t=t_2}$$

and, after adding a suitable constant we obtain a bound on e, $\|e\|_{sup} \le C$ with a constant depending only on the Lipschitz bound on $D\Phi$ and δ.

Corollary 2.2. (The maximum principle for the argument of a stationary flow) *Suppose that $v \in C(\Omega)$ is a continuous stationary solution to the 2-d Euler equation (9) on a simply-connected domain Ω such that $v \ne 0$ in Ω. Let $x_0 \in \Omega$. Let $arg\, v$ be the (continuous) argument of the vector field v defined such that $arg\, v(x_0) = 0$. Then*

$$\inf_{\partial\Omega} arg\, v \le 0 \le \sup_{\partial\Omega} arg\, v$$

with equality if and only if the direction of v is constant.

Proof. The proof anticipates the main reduction for Theorem 4. It suffices to verify the claim in a small ball. Then, without loss of generality after a rotation there is a lower bound on the first component of the velocity vector. Let Φ be the velocity potential. We introduce new coordinates $(y_1, y_2) = (x_1, \Phi(x))$ and express $x_2 = u(y)$ as a function of y. Then

$$\frac{\partial y}{\partial x} = \begin{pmatrix} 1 & 0 \\ \Phi_{x_1} & \Phi_{x_2} \end{pmatrix} \qquad \text{and} \qquad \frac{\partial x}{\partial y} = \begin{pmatrix} 1 & 0 \\ u_{y_1} & u_{y_2} \end{pmatrix}.$$

Hence

$$u_{y_2} = \Phi_{x_2}^{-1}, \quad u_{y_2} = -\frac{\Phi_{x_1}}{\Phi_{x_2}},$$

$$\frac{1}{2}(\Phi_{x_2}^2 - \Phi_{x_1}^2) = \frac{1}{2}\Big(\frac{1-u_{y_1}^2}{u_{y_2}^2}\Big)$$

and

$$\Phi_{x_1}\Phi_{x_2} = -\frac{u_{y_1}}{u_{y_2}^2}.$$

The Jacobian determinant of $x \to y$ is $\Phi_{x_2} = (u_{y_2})^{-1}$. Thus, for suitable test functions φ,

$$\int \frac{1}{2}(\Phi_{x_2}^2 - \Phi_{x_1}^2)\varphi_{x_2} + \Phi_{x_1}\Phi_{x_2}\varphi_{x_1} dx$$
$$= \int \frac{1}{2}\Big(\frac{1-u_{y_1}^2}{u_{y_2}^2}\frac{1}{u_{y_2}}\varphi_{y_2} - \frac{u_{y_1}}{u_{y_2}^2}(\varphi_{y_1} - \frac{u_{y_1}}{u_{y_2}}\varphi_{y_2}\Big)u_{y_2} dy$$
$$= \int \frac{1}{2}\frac{1+u_{y_1}^2}{u_{y_2}^2}\varphi_{y_2} - \frac{u_{y_1}}{u_{y_2}}\varphi_{y_1} dy$$

Similarly

$$\int e(\Phi)\varphi_{x_2} dx = \int e(y_2)\varphi_{y_2} dy$$

and u is a weak solution to

$$\partial_{y_1}\Big(\frac{u_{y_1}}{u_{y_2}}\Big) - \frac{1}{2}\partial_{y_2}\Big(\frac{1+u_{y_1}^2}{u_{y_2}^2} + 2e(y_2)\Big) = 0 \tag{10}$$

if and only if Φ satisfies (9).

We differentiate equation (10) with respect to y_1 and denote $v = u_{y_1}$. Then

$$\partial_i\left(a^{ij}\partial_j v\right) = 0$$

where

$$(a^{ij}) = \begin{pmatrix} \frac{1}{u_{y_2}} & -\frac{u_{y_1}}{u_{y_2}^2} \\ -\frac{u_{y_1}}{u_{y_2}^2} & \frac{1+u_{y_1}^2}{u_{y_2}^3} \end{pmatrix}$$

It is positive definite since it is symmetric, the entry a^{11} is positive and its determinant is $u_{y_2}^{-4}$. As a consequence

$$v = -\frac{\Phi_{x_1}}{\Phi_{x_2}}$$

satisfies a maximum principle, and it assumes its maximum and its minimum at the boundary. The argument of the velocity vector field (up to an additive constant) is

$$\arctan\Big(\frac{\Phi_{x_1}}{\Phi_{x_2}}\Big)$$

and the claim follows since arctan is strictly monotone. Of course this argument is not rigorous since it requires regularity of v, but it can easily be replaced by an argument using finite differences. $\square$

We also have

Lemma 2.3. *Suppose that $\Phi \in C^{0,1}(B_1)$ satisfies $\partial_2\Phi > \kappa > 0$ and (9). Then $(\partial_2\Phi, -\partial_1\Phi)$ is a stationary solution to the Euler equation (3). Moreover there is a sequence (u^j, p^j) of smooth solutions to (3) with*

$$\|u^j\|_{L^\infty} \le 2\left\|\begin{pmatrix}\partial_2\Phi \\ -\partial_1\Phi\end{pmatrix}\right\|_{L^\infty},$$
$$\inf u_1^j \ge \delta/2$$

and for any compact subset $K \subset B_1(0)$

$$\left\|u^j - \begin{pmatrix}\partial_2\Phi \\ -\partial_1\Phi\end{pmatrix}\right\|_{L^2(K)} \to 0.$$

Proof. We transform the problem by the change of dependent and independent variables as above. For the transformed problem we can regularize F, see the proof of Theorem 2. $\square$

The authors do not know whether every bounded weak stationary solution to the Euler equations (5) with a lower bound on v_1 satisfies Bernoulli's law in our formulation. The set of such solutions is however closed under weak* convergence in L^∞, but the question of approximability seems to be difficult, and we seem to be missing tools to address this delicate question.

2.2. **Traveling water waves.** The presentation here is related to the work of Constantin and Escher [7], who proved the first result in this direction assuming more regularity and required the flow to be partially periodic, with no flux boundary conditions at the bottom.

Our result applies to all flow lines, not only the free boundary. Moreover we consider local in space solutions in contrast to the global assumption in [7], and our results also apply to capillarity waves i.e. those with surface tension. The interior case has been discussed in the last section.

We consider a traveling gravity-capillarity wave in two dimensions in a moving frame of velocity c in the x_1 direction. The stationary Euler equations with a constant vertical gravitational field in a moving frame are

$$\begin{aligned} (v_1-c)\partial_{x_1}v_1+v_2\partial_{x_2}v_1 &= -\partial_{x_1}p \qquad & (v_1-c)\partial_{x_1}v_2+v_2\partial_{x_2}v_2 &= -\partial_{x_2}p-g \\ \partial_{x_1}v_1+\partial_{x_2}v_2 &= 0 \qquad & \Delta p = -\partial^2_{x_1}v_1^2-\partial^2_{x_2}v_2^2 &- 2\partial^2_{x_1x_2}(v_1v_2) \end{aligned} \tag{11}$$

in a domain U. We assume that the boundary contains (this part we call upper boundary and free boundary) the graph of a function η: We assume that there is an open interval V such that

$$U \subset \left\{(x_1,x_2)\in\mathbb{R}^2 : x_2<\eta(x_1), x_1\in V\right\}.$$

The boundary conditions at the free boundary (the graph of η) are the momentum balance

$$p=\kappa\mathcal{H}=-\kappa\partial_{x_1}\Big(\frac{\eta_{x_1}}{\sqrt{1+\eta_{x_1}^2}}\Big) \tag{12}$$

where κ is the surface tension and where $\mathcal{H}$ is the mean curvature, and the dynamic boundary condition

$$v_2=(v_1-c)\eta_x \tag{13}$$

which ensures that flow lines are tangential at the free boundary.

Since the flow is incompressible there exists a stream function ϕ so that $v_2 = -\partial_{x_1}\Phi$ and $v_1 = \partial_{x_2}\Phi + c$. The stationary Euler equations can be rewritten in terms of Φ as in the previous section,

$$\begin{aligned} \partial_{x_1}\Phi_2^2-\partial_{x_2}(\Phi_1\Phi_2)+\partial x_1 p =&0 \\ -\partial_{x_1}(\Phi_2\Phi_1)+\partial_{x_2}(\Phi_1^2)+\partial_{x_2}p =&-g \end{aligned} \tag{14}$$

We define the energy density

$$e=\frac{1}{2}(|\Phi_1|^2+|\Phi_2|^2)+p+gx_2$$

and the stationary Euler equations become

$$\begin{aligned}\frac{1}{2}\partial_{x_1}\left(\Phi_2^2-\Phi_1^2\right)+\partial_{x_2}(\Phi_1\Phi_2)=&\partial_{x_1}e\\ \partial_{x_1}(\Phi_2\Phi_1)+\frac{1}{2}\partial_{x_2}\left(\Phi_1^2-\Phi_2^2\right)=&\partial_{x_2}e.\end{aligned}\tag{15}$$

By the dynamic boundary condition Φ is constant on the free boundary. It is determined up to a constant which we choose so that $\Phi=0$ at the free boundary. Again e is a function of Φ and it is constant at the free boundary. We denote this constant by e_0.

The dynamic boundary condition becomes $\Phi=0$ and the momentum balance is

$$\frac{1}{2}|\nabla\phi|^2+g\eta=\kappa\partial_{x_1}\left(\frac{\partial_{x_1}\eta}{\sqrt{1+(\partial_{x_1}\eta)^2}}\right)-e_0.\tag{16}$$

We assume $v_1<c$, i.e. a wave is faster than the supremum of the velocity of the particles. This implies that $\partial_{x_2}\phi\le-\kappa<0$. Then the velocity potential Φ satisfies again (9) in a set where the free boundary is given by $x_2=\eta(x_1)$. The boundary condition is

$$e-gx_2-\frac{1}{2}|\nabla\Phi|^2=\kappa\mathcal{H}.$$

We rewrite the problem as

$$\begin{aligned}\frac{1}{2}\partial_{x_2}(\Phi_2^2-\Phi_1^2-2e(\Phi))+\partial_{x_1}(\Phi_1\Phi_2)=0\text{ in }x_2<\eta(x_1)\\ \frac{1}{2}|\nabla\Phi|^2+g\eta=e+\kappa\partial_{x_1}\left(\frac{\eta_{x_1}}{\sqrt{1+\eta_{x_1}^2}}\right)\text{ on }x_2=\eta(x_1)\\ \Phi=0\text{ on }x_2=\eta(x_1).\end{aligned}\tag{17}$$

We introduce new coordinates $(y_1,y_2)=(x_1,\phi(x))$ and express $x_2=u$ as a function of y. Then we obtain the same equation as above for the bulk

$$\partial_{y_1}\left(\frac{u_{y_1}}{u_{y_2}}\right)-\frac{1}{2}\partial_{y_2}\left(\frac{1+u_{y_1}^2}{u_{y_2}^2}\right)=\partial_{y_2}e(y_2),$$

combined with the boundary condition

$$\frac{1}{2}\frac{1+u_{y_1}^2}{u_{y_2}^2}+e_0=gu-\kappa\partial_{y_1}\left(\frac{u_{y_1}}{\sqrt{1+u_{y_1}^2}}\right)\tag{18}$$

Theorem 2.4. *Let $g,\kappa\in\mathbb{R}$. Suppose that the velocity field Φ is the stream function of a bounded solution which satisfies* (9) *in an open set U as above together with the boundary conditions* (12) *and* (13)*. We assume that the horizontal velocity $c+\partial_{x_2}\Phi$ is below c. Then the flow lines and the free boundary are analytic.*

Proof. Let $w=\partial_{y_1}u$. It satisfies formally

$$\begin{aligned}\partial_i a^{ij}\partial_j w=&0 \qquad \text{for } y_2>0\\ a^{2j}\partial_j w-gw=&-\partial_{x_1}(b\partial_{x_1}w) \qquad \text{for } y_2=0\end{aligned}$$

where a^{ij} is the positive definite matrix

$$\begin{pmatrix}-\frac{1}{u_{y_2}} & \frac{u_{y_1}}{u_{y_2}^2}\\ \frac{u_{y_1}}{u_{y_2}^2} & -\frac{1+u_{y_1}^2}{u_{y_2}^3}\end{pmatrix}$$

and

$$b = \kappa\Big(1 + (\frac{u_{y_1}}{u_{y_2}})^2\Big)^{-3/2}.$$

If $\kappa = 0$ then the assertion follows from the boundary version with Neumann boundary conditions of the statements of the last section. We postpone the proof for the case $\kappa \neq 0$ to the end of the paper. □

2.3. **The obstacle problem.** We consider viscosity solutions to

$$\min\{u, 1 - \Delta u\} = 0$$

in an open set which is a concise formulation of the obstacle problem. Alternative formulations are

$$u \geq 0, \Delta u \geq 1, u(\Delta u - 1) = 1$$

as well as the variational characterization as local minimizers of

$$\int \frac{1}{2}|\nabla u|^2 + u dx$$

subject to the constraint $u \geq 0$. Solutions have locally Lipschitz continuous derivatives. The contact set is defined to be $K = \{x : u(x) = 0\}$. Its boundary is the free boundary. The regular part consists of points x_0 in the free boundary such that in a neighborhood outside K possibly after a rotation

$$\partial^2_{x_n x_n} u \geq \kappa > 0.$$

On K one has $\nabla u = 0$.

Theorem 5. *The regular part of the free boundary is analytic.*

This is well known due to the work of Caffarelli and coworkers [2] who prove that $u \in C^{2,\alpha}$ on the closure of the positivity set near a regular free boundary point. The regular free boundary is analytic by the work of Kinderlehrer and Nirenberg [13]. Here we want to show that the obstacle problem fits into the context of this paper.

Proof. Locally we may assume that the free boundary is the graph of a Lipschitz function $x_n = \eta(x')$, and that the contact set is below the graph. Let $v = \partial_n u$. Above the graph it is harmonic. It vanishes at the boundary.

Moreover, if $\phi \in C_0^\infty$ with sufficiently small support, then

$$\begin{aligned}\int \nabla v \nabla \phi dx &= -\int \nabla u \partial_n \nabla \phi dx \\ &= \int_{x_n > \eta} \partial_n \phi dx \\ &= -\int_{\mathbb{R}^{n-1}} \phi(x', \eta(x')) dx'\end{aligned} \tag{19}$$

by an application of Fubini's theorem.

Then v is harmonic in the positivity set. It is Lipschitz continuous and the level surfaces are Lipschitz graphs. It satisfies

$$v = 0$$

and the conormal boundary condition

$$\partial_\nu v = -1$$

at the boundary where ν denotes the exterior normal.

Now let $y_i = x_i$, $y_n = v$ and $w(y) = x_n$. We change coordinates in equation (19):

$$\begin{aligned} \int \nabla_x v \cdot \nabla_x \varphi dx &= \int \Big[\sum_{i=1}^{n-1} (-\frac{w_{y_i}}{w_{y_n}} (\varphi_{y_i} - \frac{w_{y_i}}{w_{y_n}} \varphi_{y_n}) + \frac{1}{w_{y_n}^2} \varphi_{y_n} \Big] w_{y_n} dy \\ &= \int - \sum_{i=1}^{n-1} w_{y_i} \varphi_{y_i} + \frac{1 + \sum_{i=1}^{n-1} w_{y_i}^2}{w_{y_n}} \varphi_{y_n} dy \end{aligned}$$

Thus w is a weak solution to

$$\sum_{i=1}^{n-1} \partial_i^2 w - \partial_n \frac{1 + \sum_{j=1}^{n-1} |\partial_j w|^2}{\partial_n w} = 0$$

with boundary condition

$$\frac{1 + \sum_{j=1}^{n-1} |\partial_j w|^2}{\partial_n w} = 1.$$

This is a boundary analogue of Theorem 4 and it can be proven along the same lines. Alternatively we may write

$$w = x_n + \hat{w},$$

check that

$$\sum_{i=1}^{n-1} \partial_i^2 \hat{w} + \partial_n \frac{\partial_n \hat{w} - \sum_{j=1}^{n-1} \partial_j \hat{w}^2}{1 + \partial_n \hat{w}} = 0$$

$$\frac{\partial_n \hat{w} - \sum_{j=1}^{n-1} |\partial_j \hat{w}|^2}{1 + \partial_n w} = 0.$$

An even extension of $\hat{w}$ is a weak solution of an elliptic equation of the type considered in Theorem 4 with a discontinuity at $y_n = 0$ and the claim follows from the proof pf Theorem 4 below. □

2.4. **Harmonic maps.** Let (M^n, g) (N^k, h) be Riemannian manifolds with metrics $\sum g_{lm} dx^l dx^m$ and $\sum h_{ij} du^i du^j$. Let u be a C^1 map from M^n to N^k. A map $u : M^n \to N^k$ is called harmonic map if its Dirichlet integral is a local extremal of the map.

The preimage $u^{-1}(\{y\})$ is called the level set to the level y. As a consequence of Theorem 3 we have

Theorem 2.5. *Assume (M^n, g) is a real analytic Riemannian manifold, (N^k, δ) is a C^2 manifold with a C^1 metric δ and $k < n$. Then all level sets of the continuous harmonic map u outside its critical points are real-analytic submanifolds.*

Proof. It is well known that continuous harmonic maps are as smooth as the data permit, which is here $C^{1,s}$ for all $0 < s < 1$, see [17] . Let $x \in M^n$, $y = u(x) \in N^k$. We choose local coordinates near x in M, denote the Laplace-Beltrami operator of M by Δ_M, and denote the Christoffel symbols of the Levi-Civita connection on N by Γ_l^{ij}. The Christoffel symbols are continuous. In these coordinates the harmonic map equations i.e. the Euler-Lagrange equations of the Dirichlet integral can be written as

$$\Delta_M u_k = \sum_{\alpha,i,j} \Gamma_k^{ij}(u) \partial_\alpha u_i \partial_\alpha u_j.$$

The statement in local coordinates follows now from Theorem 1. □

2.5. **Nonlinear Schrödinger equations.** Nonlinear Schrödinger equations of the type

$$i\frac{\partial u}{\partial t} + \Delta u = f(|u|)u, \tag{20}$$

where $u : \Omega \to \mathbb{C}$, $\Omega \subset \mathbb{R}^n$, $p \geq 0$. This appears in many physical models: internal gravity waves, ferromagnetism, nonlinear optics and plasma theory, see [6]. For $f = |u|^2$ and '+' (20) is the Ginsburg-Landau equation. It also appears in nonlinear optics. Stationary solutions of (20) as well traveling wave type solutions satisfy an elliptic system (21). The nodal set $u^{-1}(\{0\})$ is of particular interest.

We do not require that f in (20) is analytic. However, as a consequence of Theorem 2 we see that nodal surfaces of stationary solutions to (20) are real analytic.

Theorem 2.6. *Let u be a bounded complex valued stationary solution of* (20) *in a domain $\Omega \subset \mathbb{R}^n$:*

$$\Delta u + f(|u|)u = 0 \tag{21}$$

with some continuous function f. Let Γ be a nodal set of u, $z \in \Gamma$ and real rank $Du(z) = 2$. Then in a neighborhood of z Γ is a $n-2$-dimensional real analytic surface. If $D|u|^2 \neq 0$ and $r > 0$ then

$$\{x : |u(x)| = r\}$$

is a real analytic surface of dimension $n - 1$.

Proof. The first statement is an immediate consequence of Theorem 3. For the second statement we have to check the proof. First standard elliptic estimates imply that the function u is in C^1. We fix a point x_0 with $|u(x_0)| = r$. Without loss of generality we may assume that $u(x_0) = r$. Let

$$w(x) = \Im u(x)$$

Without loss of generality we assume that $\partial_{x_n}|u|^2(x_0) \neq 0$. We choose

$$y_i = x_i \qquad \text{if } 1 \leq i < n \qquad y_n = |u|, \qquad v(y) = x_n.$$

A tedious calculation leads to the elliptic system of equations for v and w and an application of Theorem 1 implies the claim. We leave the tedious and instructive calculation to the reader.

□

2.6. **Free boundary problems of Grad-Mercier type.** We consider a free boundary problem of the type arising from variational inequalities. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and $u \in W^{2,p}(\Omega)$, satisfying,

$$\begin{cases} -\Delta u + g(u) = 0, & \text{in } \Omega \\ u = \text{ (unknown) constant } > 0 & \text{on } \partial\Omega \\ \int_{\partial\Omega} \frac{\partial u}{\partial n} = -1 & \end{cases} \tag{22}$$

where $\Gamma = \{x \in \Omega, u(x) = 0\}$ is a free boundary. Equation (22) is an equivalent form of the Grad-Mercier equation of equilibrium of plasma, see [11]. The function g is a bounded but generically discontinuous function. At 0 function g has a discontinuity of the first kind. A typical example, $g(t) = t$ for $t > 0$ and $g(t) > C > 0$ for $t < 0$, see [11].

As a consequence of Theorem 4 we have

Theorem 2.7. *Suppose that* 0 *is a point in the free boundary* Γ *and* $\nabla u(0) \neq 0$. *Then* Γ *is an analytic surface in a neighborhood of* 0.

Assume additionally in the theorem that $n = 2$, Ω is a convex domain and u is a locally stable solution of a corresponding variational functional. Under such assumptions Cabre and Chanillo [5] proved that a solution u of a variational problem has a single critical point. Thus as a corollary we have: under the above additional conditions to Theorem 4 the free boundary Γ is a closed analytic curve.

2.7. **Examples and counter examples for C^2 regularity.** Let u be a solution of a semi-linear equation

$$\Delta u = f(u) \tag{23}$$

in a domain $U \subset \mathbb{R}^n$. Suppose $f \in C^s$, $0 \leq s \leq 1$. For $0 < s < 1$ the inner regularity of u follows from the standard Schauder estimates:

$$u \in C^{2+s}$$

For $s = 0$ standard estimates show that u lies locally in the Besov $B^2_{\infty,\infty}$. However, we show for some classes of solutions of equation (23) for $s = 0$ the implication $u \in C^2$ holds true.

Let $u \in B^2_{\infty,\infty}(D)$ and $\nabla u(x_0) = 0$, $x_0 \in D$. We say that x_0 is a Morse point if u has a second differential at x_0 of the Morse type, i.e., there exists a quadratic form $q(x)$ with non-zero eigenvalues such that

$$q(x) - u(x - x_0) = o(|x|^2).$$

Lemma 2.8. *Let* u *be a solution of* (23). *Suppose* $f \in C^0$. *If all critical points of* u *are of Morse type then* $u \in C^2$.

Proof. Critical points of Morse type are isolated. Let $B \subset D$ be a ball in D and $|u| > \delta > 0$ in B. We assume that $\nabla u \neq 0$ in B and set $e_1(x) = \nabla u/|\nabla u|$

$$\Gamma_t = \{x \in B,\, u(x) = t\}.$$

Since by assumption $\nabla u \neq 0$ the level surfaces are regular, and they depend continuously on t. By Theorem 4 the level sets Γ_t are locally uniformly analytic. Let $H(x)$ be the mean curvature of the level set $\Gamma_{u(x)}$ at the point x. It is a continuous function of x. Now

$$\Delta u = \sum_{i,j=1}^{n} e_{1,i} e_{1,j} \partial_{ij} u + H \partial_{e_1} u = f(u)$$

and, since all terms besides the first are continuous, the first term is continuous as well, and all second derivatives are continuous.

It remains to prove the theorem in neighborhoods of the critical points of the function u. Suppose that $0 \in U$ is a critical point of u. Define $u_a = u(ax)/a^2$, $a > 0$. From the assumptions of the theorem

$$u_a \to U$$

in $C^1(B_1)$ as $a \to 0$, where $B_1 \subset \mathbb{R}^n$ is a unit ball and U is a quadratic form with non-zero eigenvalues. Therefore ∇u_a is bounded away from 0 in the spherical shell

$B_1 \setminus B_{1/2}$ for all sufficiently small a. Thus $|D^2 u_a|$ is bounded in $B_1 \setminus B_{1/2}$ for all small a and $u_a \to U$ in C^1 and

$$D^2 u_a \to D^2 U$$

weak* in L^∞.

The level sets are uniformly analytic, and the same is true for the gradient restricted to the level sets. As a consequence second order derivatives containing one tangential direction converge uniformly. Since we can solve the equation for the remaining second order derivative $D^2 u_a \to D^2 U$ uniformly in $B_1 \backslash B_{1/2}$.

□

In [25] H. Shahgholian raised a question: Is it true that a function $u \in C^1$ which satisfies (23) with a continuous non-linearity f is automatically a C^2 function? Below we show that in Theorem 2.8 one can not drop the assumptions on the critical points of the solution.

Lemma 2.9. *There exists a continuous function f with values in $[-1, 0]$ and $u \in B^2_{\infty,\infty}(B_1(0))$ which satisfy*

$$\Delta u = f(u) \qquad \text{in } B_1(0) \tag{24}$$

and $u(x) = x_1 x_2$ for $|x| = 1$ which has no bound of the type $C|x|^2$. More precisely $u(0) = Du(0) = 0$ and

$$\sup_x \frac{u(x)}{|x|^2} = \infty.$$

Proof. Let Q be the first quadrant of $\mathbb{R}^2$:

$$Q = \{x \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}.$$

Let w be a solution in Q of the Dirichlet problem

$$\begin{cases} \Delta w = -1, & \text{in } Q \cap B_1(0) \\ w = 0 & \text{on } \partial Q \cap B_1(0) \\ w = x_1 x_2 & \text{on } Q \cap \partial B_1(0) \end{cases} \tag{25}$$

One easily checks that

$$\Delta\{[(-\ln(x_1^2 + x_2^2)) + 1]x_1 x_2\} = -2\frac{x_1 x_2}{|x|^2}$$

and that this function satisfies the boundary conditions. The solution u to

$$\Delta \widetilde{u} = -1 + \frac{2x_1 x_2}{x_1^2 + x_2^2}$$

with boundary data $\widetilde{w}(x) = 0$ if $|x| = 1$ or $x_1 = 0$ or $x_2 = 0$ is given by

$$\widetilde{u}(x) = \frac{4}{\pi} \sum_{j=2}^{\infty} \frac{1}{(2j)^3} \Big(\frac{\mathrm{Re}(x_1 + ix_2)^{2j}}{|x|^{2j-2}} - \mathrm{Re}(x_1 + ix_2)^{2j} \Big)$$

which is easily seen to be twice differentiable. Thus

$$w - [(-\ln(x_1^2 + x_2^2)) + 1]x_1 x_2 \in C_b^2 \tag{26}$$

has bounded second order derivatives.

We will choose functions f with values between 0 and 1. By the maximum principle any solution (which will in general be non unique) is bounded from below by the positive harmonic function $x_1 x_2$, and from above by w. For any sequence

of functions f converging to the negative of the Heaviside function the solutions converge to w. Hence, for $k \in \mathbb{N}$ there exists f_k such that

$$\sup \frac{u_k}{|x|^2} \geq k^2.$$

We define

$$f(u) = \max \frac{f_k(u)}{k}.$$

The solution u satisfies

$$\limsup_{x \to 0} \frac{u}{|x|^2} \geq \sup_k \limsup_{x \to 0} \frac{u_k(x)}{k|x|^2} = \infty.$$

Suppose that the function f is extended oddly on $\mathbb{R}$: $f(t) = -f(-t)$. We chose an odd extension of the function u over the coordinate axis on $\mathbb{R}^2$. The extended u will satisfy the equation $\Delta u = f(u)$ on B_1. The function u is continuously differentiable with $\nabla u(0) = 0$. Due to the lower bound u cannot be twice differentiable at 0. $\square$

3. Singular integral type estimates

We consider a linear elliptic system

$$\partial_i a^{ij}_{kl} \partial_j u^l = \partial_i F^i_k$$

with $1 \leq i, j \leq n$ and $1 \leq k, l \leq N$ which we write in divergence form. The map $F \to \nabla u$ is a Calderón-Zygmund operator and the following estimates are standard.

Proposition 3.1. *Suppose that u has derivatives in $L^1 \cup L^q$ for some $q < \infty$ and*

$$\partial_i a^{ij}_{kl} \partial_j u^l = \partial_i F^i_k.$$

Then

$$\|Du\|_{L^p} \leq c_p \|F\|_{L^p}.$$

Moreover, if the derivatives of u grow at most linearly and $0 < s < 1$ then

$$\sup_{x \neq y} \frac{|Du(x) - Du(y)|}{|x-y|^s} \leq c \sup_{x \neq y} \frac{|F(x) - F(y)|}{|x-y|^s}.$$

For $n_2 = 1$ there is a variant to the first Hölder estimate: We may restrict to difference quotients in the first $n-1$ variables. We recall the notation $(x', x'') \in \mathbb{R}^{n-1} \times \mathbb{R}$.

Proposition 3.2. *Suppose that*

$$\partial_i a^{ij}_{kl} \partial_j u^l = \partial_i F^i_k$$

in $\mathbb{R}^n$. If

$$|\nabla u| \leq c(1 + |x|)$$

then

$$\sup_{x' \neq y', x''} \frac{|Du(x', x'') - Du(y', x'')|}{|x' - y'|^s} \leq c \sup_{x' \neq y', x''} \frac{|F(x', x'') - F(y', x'')|}{|x' - y'|^s}$$

Proof. We formulate the crucial result for singular integral operators.

Lemma 3.3. *Let $0 < s < 1$. We denote a point in $\mathbb{R}^n$ by $(t,x) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Let C_0^s be the homogeneous Hölder space of Hölder continuous functions with compact support and let*

$$T : C_0^s(\mathbb{R}^n) \to L^\infty$$

be a partial convolution operator with integral kernel $k(\tau, t, x)$ (i.e.

$$Tf(t,x) = \int k(\tau, t, x-y) f(\tau, y) dy d\tau$$

under suitable assumptions on the support) which satisfies

$$|k(\tau,t,x)| \le \frac{c}{|(\tau - t, x)|^n},$$

$$|k(\tau,t,x) - k(\tau,t,y)| \le c \frac{|x-y|}{\min\{|(\tau-t,x)|, |(\tau-t,y)|\}^{n+2}},$$

and, if $t \neq \tau$ and $R > 0$

$$\left| \int_{B_R^{\mathbb{R}^{n_1}}(0)} k(\tau,t,x)\, dx \right| \le cR^{-1} \tag{27}$$

Then T has a unique extension to $T : L_t^\infty C_x^s \to L_t^\infty C_x^s$, for $0 < s < 1$ and it satisfies

$$\sup_{t, x \neq y} \frac{|Tf(t,x) - Tf(t,y)|}{|x-y|^s} \le c \sup_{t, x\neq y} \frac{|f(t,x) - f(t,y)|}{|x-y|^s}$$

Remark 3.4. *We denote the seminorm by C_*^s,*

$$\|f\|_{C_*^s} = \sup_{t,x\neq y} \frac{|f(t,x) - f(t,y)|}{|x-y|^s}.$$

We apply this to the Calderón-Zygmund operator

$$f \to \partial_{ij}^2 u.$$

As a homogeneous convolution operator of Calderón-Zygmund operator type with smooth kernel it always satisfies the first two conditions. The cancellation condition follows from the fact that the kernel is the derivative in one of the directions, unless $i = j = n$, but in that case we see that it is the sum of second derivatives by using the equation (the kernel is a solution to the equation with respect to the first variable, and to the adjoint equation with respect to the second variable away from the diagonal). Proposition 3.2 follows from Lemma 3.3 which we turn to next. □

Proof of Lemma 3.3. Let $x_1 \in \mathbb{R}^{n-1}$ with $|x_1| = 1$. The claim follows from the estimate

$$\left| \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-1}} (k(\tau, 0, x_1 - y) - k(\tau, 0, -y)) f(\tau, y) dx\, d\tau \right| \le c \|f\|_{L_t^\infty \dot{C}_x^s} \tag{28}$$

by translation in t, x and scaling. It is part of the assertion that the integral exists. We turn to the proof of (28). In a first step we show that it suffices to prove this estimate under the additional assumption $f(t,0) = 0$.

Step 1: We fix $\eta \in C_0^\infty(\mathbb{R}^{n-1})$, supported in the unit ball $B_1 \subset \mathbb{R}^{n-1}$ and identically 1 in a ball of radius 1/2. Then by the cancellation condition and the pointwise condition on the kernel

$$\left| \int k(\tau,0,x)\eta(x/R)dx \right| = \left| \int_{B_{R/2}} k(\tau,0,x)dx \right| + \left| \int_{B_R \setminus B_{R/2}} k(\tau,0,x)\eta(x)dx \right| \tag{29}$$
$$\leq CR^{-1}.$$

Let $f \in L^\infty_t \dot C^s$ (where $\dot C^s$ is the homogeneous Hölder space with semi norm $\sup \frac{|u(x)-u(y)|}{|x-y|^s}$) with compact support. We define for $\rho \geq 1$

$$R(t) = \rho(1+|f(t,0)|)(1+|t|^2)$$

and

$$g(t,x) = f(t,0)\eta(x/R(t)).$$

Then by (29)

$$\left| \int k(\tau,0,x) g(\tau,x)dx \right| \leq c[\rho(1+|\tau|^2)]^{-1}$$

and hence

$$|Tg(0)| + |Tg(x_1)| \leq c\rho^{-1}$$

independent of f. It tends to 0 as $\rho \to \infty$. As a consequence it suffices to prove the key inequality (28) for f which satisfies $f(t,0) = 0$, which we assume from now on.
Step 2: Let $h(t,x) = \eta(2(x-x_1))f(t,x_1)$. Then, using the cancellation condition, for $x = 0$ and $x = x_1$,

$$\left| \int_{\mathbb{R}^{n-1}} k(\tau,0,x-y)h(\tau,y)dy \right| \leq c|f(t,x_1)|$$

for $|\tau| \leq 2$. Let $\chi = \chi_{[-1,1]}(t)$ and $f_2 = f - \chi h$. It vanishes at $x = 0$ and at $x = x_1$ for $|t| \leq 1$. Then

$$\begin{aligned} \left| \int k(\tau,-y) f_2(\tau,y)dy \right| \leq & c \int (|\tau|+|y|)^{-n}|y|^s dx \sup_z \frac{|f(\tau,z)|}{|z|^s} \\ \leq & c|\tau|^{s-1} \sup_{x\neq y} \frac{|f(\tau,x)-f(\tau,y)|}{|x-y|^s} \end{aligned}$$

which we use for $|t| \leq 1$. The same bound holds at $x = x_1$. Together

$$|(T(\chi f)(0,x_1) - (T\chi f)(0,0)| \leq c \sup_{|t|\leq 1} \sup_{x\neq y} \frac{|f(t,x)-f(t,y)|}{|x-y|^s}.$$

We turn to $|\tau| \geq 1$ and estimate

$$\begin{aligned} &\left| \int_{\mathbb{R}^{n-1}} (k(\tau,0,-y) - k(\tau,0,x_0-y)) f(t,y)dy \right| \\ &\qquad \leq \int_{\mathbb{R}^{n-1}} |y|^s(|\tau|+|y|)^{-n-1} dy \sup_{x\neq y} \frac{|f(\tau,x)-f(\tau,y)|}{|x-y|^s} \\ &\qquad \leq c|\tau|^{-2+s} \|f(t,.)\|_{\dot C^s(\mathbb{R}^{n-1})}. \end{aligned}$$

We integrate that estimate over $\mathbb{R}\backslash(-1,1)$ and arrive at (28). The proof extends to $f \in L^\infty \dot C^s$, and this allows to extend the assertion to the weak (distributional) closure of $\dot C^s$ in that space, which is the whole space. $\square$

We may weaken the assumption at least in the scalar case. Consider

$$\partial_{x_i} a^{ij}(t)\partial_{x_j} u = \partial_i F^i$$

with measurable uniformly bounded and elliptic coefficients a^{ij}. We use the index 0 for the coordinates corresponding to t. We use the same convention for the Green's function as for the kernel above.

Lemma 3.5. *The Green's function satisfies*

$$|\partial_x^\alpha \partial_t^l \partial_\tau^k g(\tau,t,x)| \le c(|x|+|t-\tau|)^{2-n-|\alpha|-l-k}$$

for $k,l \le 1$ and all multiindices α.

Proof. There exists a unique Green's function G which satisfies

$$G((x,t);(y,\tau)) = g(\tau,t,x-y) = g(t,\tau,y-x)$$

for some function g,

$$|g(t,s,x)| \le c(|x|+|s-t|)^{2-n}$$

if $n \ge 3$, see Grüter and Widman [12]. Moreover we show that with $4r^2 = |x|^2 + |t-s|^2$

$$r^{-n}\int_{B_r(x,t)} |\nabla_{\tau,x} g(\tau,t,y)|^2 dyds \le cr^{1-n}.$$

By a scaling argument it suffices to consider a ball of radius 1 around (t_0,x_0) with $|t_0-s|^2+|x_0-y|^2=4$, and hence to bound

$$|\partial^\alpha \partial_t^l u(0,0)| \le c_\alpha \|u\|_{L^2(B_1(0,0))}$$

for $l \le 1$ and a solution u to the homogeneous problem in B_2. Recursive L^2 estimates imply the estimates for

$$\partial_x^\alpha \partial_t^l u$$

in L^2 for $l \le 1$. This implies pointwise estimates for $\partial_x^\alpha u$ in terms of the L^2 norm. We rewrite

$$\partial_t g^{0j}\partial_j \partial^\alpha u = -g^{i0}\partial_t\partial_i\partial^\alpha u - g^{ij}\partial^2_{ij}\partial^\alpha u.$$

The second term on the right hand side is bounded. The first term is in $L^2_t L^\infty_x$. Thus

$$a^{0j}\partial_j\partial^\alpha u$$

is bounded and hence $\partial_t\partial^\alpha g$ is bounded in terms of the L^2 norm. The Green's function is symmetric, and the remaining estimate for $\partial_t\partial_\tau\partial_x^\alpha g$ follows by repeating the previous arguments. □

Proposition 3.6. *Under these assumptions Proposition 3.2 holds.*

Proof. It suffices to check that the kernel bounds of Lemma 3.5 are sufficiently strong for the proof of Proposition 3.2. Only the cancellation condition is not obvious. As above the cancellation condition is immediate for $\partial^2_{x_ix_j} g(\tau,t,x)$ and for $\partial^2_{x_it} g(\tau,x)$ and $\partial^2_{\tau x_i} g(\tau,t,x)$ since then the kernel contains a derivative in x direction. Only $\partial_t\partial_s g(s,t,x)$ requires additional considerations.

The Green's function is a solution to the homogeneous problem away from the diagonal. Let

$$u(t,x) = \partial_\tau g(\tau,t,x).$$

It satisfies

$$|u(t,x)| \le c(|t-\tau|+|x|)^{1-n}$$

and

$$|\partial_t u(t,x)| \le c(|t-\tau|+|x|)^{-n}.$$

Moreover it is a solution to the homogeneous equation away from $t=\tau$ and $x=y$ hence

$$\partial_t(a^{00}\partial_t u) = -\partial_\alpha \partial_t a^{0\alpha} u - \sum_{\alpha,\beta\ge 1} \partial_\alpha a^{\alpha\beta}\partial_\beta u - \sum_{\alpha \ge 1} \partial_\alpha a^{\alpha,0}\partial_t u \tag{30}$$

Let $R>0$. We want to prove for $t\neq\tau$ that

$$\left| \int_{B_R} u_t(t,x)dx \right| \le cR^{-1}$$

which is trivial for $R \le |t-\tau|$. Let $t > \tau + R$ without loss of generality. Then, by the previous formula and an application of the divergence theorem

$$\begin{aligned}\left|\int_{B_R} a^{00} u_t(t,x)\,dx\right| \le & \left|\int_{B_R} \partial_\alpha a^{0\alpha} u\,dx\right| + \left|\int_{\partial B_R}\int_t^\infty (R+|s-\tau|)^{-n}\,ds\right| \\ \le & CR^{-1}.\end{aligned}$$

□

We conclude this section with some existence and uniqueness statements.

Lemma 3.7. *Let $F^i_k \in C^s$ and $f \in L^{n/s}$ be supported in the unit ball. Then there exists a unique solution u to*

$$\partial_i a^{ij}_{kl}(x_n)\partial_j u^l = \partial_i F^i_k + f_k$$

which satisfies

$$\|Du\|_{\dot C^s(\mathbb{R}^n)} \le c\left(\|F\|_{\dot C^s(\mathbb{R}^n)} + \|f\|_{L^{n/s}}\right)$$

and

$$u \to 0 \qquad \text{as } x\to\infty$$

if $n\ge 3$ or

$$u(0)=0$$

and

$$|u| \le c(1+|x|^\varepsilon)$$

if $n=2$. Similarly, if $a^{ij}(x_n)$ is bounded, measurable and uniformly elliptic, $F^i \in C^s_*$ with compact support and $f\in L^{n/s}$ with support in $B_1(0)$ then there is a unique solution u to

$$\partial_i a^{ij}\partial_j u = \partial_i F^i + f$$

which satisfies

$$\|Du\|_{C^s_*} \le c\left(\|F\|_{C^s_*} + \|f\|_{L^\infty}\right)$$

and

$$u \to 0 \qquad \text{as } x\to\infty$$

if $n\ge 3$ or

$$u(0)=0$$

and

$$|u| \le c(1+|x|^\varepsilon)$$

if $n=2$.

Proof. The convolution with the fundamental solution gives a function which satisfies the homogeneous estimates. The solution is unique up to the addition of an affine function. The condition fixes this affine function. The estimates in the inhomogeneous norm are a consequence of the bounds of the fundamental solution. The same arguments apply to the second part. $\square$

4. Partial holomorphy for linear systems

We consider the elliptic system of equations

$$\partial_i a^{ij}_{kl} \partial_j u^l = \partial_i F^i_k + f_k \tag{31}$$

on $B_2(0) \subset \mathbb{R}^n$ under the assumption that the constant coefficients are elliptic and $u \in C^{1,s}(\overline{B_1(0)})$ for some $s > 0$.

We will consider partially analytic functions F and f which are given as partially holomorphic functions in a partially complex domain. Given $\delta > 0$ we define the complexified unit ball

$$B_\delta = \Big\{ (x' + iy', x'') : |y'| \le \delta(1 - |x|^2)^3_+ \Big\}.$$

Definition 4.1. *Let $\delta > 0$ and $0 < s < 1$. We define the norms*

$$\|F\|_{C^s_\delta} = \sup_{\theta \in \mathbb{R}^{n_1}, |\theta| \le \delta} \|F(x' + i\theta(1 - |x|^2)^3_+, x'')\|_{C^s(B_2(0))},$$

$$\|F\|_{C^s_{*,\delta}} = \sup_{\theta \in \mathbb{R}^{n_1}, |\theta| \le \delta} \|F(x' + i\theta(1 - |x|^2)^3_+, x'')\|_{C^s_*(B_2(0))},$$

and

$$\|f\|_{L^\infty_\delta} = \|f\|_{L^\infty(B_\delta) \cap L^\infty(B_2(0))}.$$

The corresponding function spaces are the spaces of functions for which these norms are finite, and which are holomorphic in $z' = x' + iy'$.

Holomorphy is equivalent to the Cauchy-Riemann equations. The Cauchy integral formula implies estimates for derivatives. It will be useful later on that we allow x in the ball of radius 2. Let $F \in C^s_\delta$. We define

$$F^0(x' + iy', x'') := F(x', x'').$$

Proposition 4.2. *Suppose that $n \ge 2$, and that the coefficients a^{ij} are elliptic. There exists $\delta_0 > 0$ so that the following holds. Suppose that $0 < \delta \le \delta_0$,*

$$F \in C^s_\delta, f \in L^\infty_\delta$$

and that $u^0 \in C^{1,s}(B_2)$ satisfies

$$\partial_i a^{ij}_{kl} \partial_j u^{0,l} = \partial_i F^i_k + f_k.$$

Then there exists a unique partially holomorphic extension to B_δ with

$$\|u - u^0(x)\|_{C^{1,s}} \le c \left(\delta \|u^0\|_{C^{1,s}} + \|F - F^0\|_{C^s_\delta} + \|f - f^0\|_{L^\infty_\delta} \right).$$

Proof. We consider only scalar equations of the type

$$\partial_l a^{lk} \partial_k u = f$$

to simplify the exposition. The case of systems requires not more than obvious modifications. We treat weak derivatives on a formal level. This can be justified by testing by a function in $\mathbb{R}^{n_2}$, integrating, and checking that the derivatives with respect to the first n_1 derivatives always exist.

If u with $Du \in C^s_{\delta_1}$ is a partially holomorphic solution then the function (with $|\theta| \le \delta$)

$$u^\theta(x) = u(x' + i\theta(1-|x|^2)^3_+, x'')$$

satisfies

$$L^\theta u^\theta := \partial_{X_i} a^{ij} \partial_{X_j} u^\theta = f_\theta \tag{32}$$

where (with $\phi = (1-|x|^2)^3_+$)

$$\partial_{X_j} = \partial_{x_j} - \frac{i\partial_{x_j}\phi}{1 + i\theta\cdot\nabla\phi}\theta_l \partial_{x_l}. \tag{33}$$

Equation (32) is elliptic provided δ_0 is sufficiently small. It will be useful to consider

$$v^\theta = u^\theta(x) - u(x)$$

which we extend by 0 outside the unit ball. The function v^θ has compact support, and it is a solution to

$$\partial_i a^{ij}\partial_j v^\theta = \partial_i(a^{ij} - a^{ij}_\theta)\partial_j v^\theta + \partial_i(F^i_\theta - F^i) + (f_\theta - f) - \partial_i(a^{ij}_\theta - a^{ij})\partial_j u \tag{34}$$

on $\mathbb{R}^n$ with v^θ and the right hand side supported in $B_1(0)$. A simple fixed point argument shows that this equation has a unique solution which satisfies

$$\|\nabla v^\theta\|_{C^s} \le c\Big(\|F^i_\theta - F^u\|_{C^s} + \|f_\theta - f\|_{L^\infty} + |\theta|\|Du\|_{C^s}\Big)$$

as well as

$$\nabla v^\theta(x) \to 0 \qquad \text{as } x \to \infty \tag{35}$$

and

$$v^\theta(x_0) = 0 \qquad \text{at a chosen point } x_0. \tag{36}$$

This yields the estimate of Proposition 4.2 and it remains to verify existence of a partially holomorphic solution.

For the existence argument we consider $\theta \in \mathbb{R}^{n_1}$ with $|\theta| < \delta$ and define v^θ (and hence $u^\theta = v^\theta + u$) as solution to (34) with the normalizing conditions (35) and (36) where x_0 is a point outside the ball of radius 2 . We obtain a family of solution to (32). Since we construct the solution via a fixed point argument respectively via an implicit function theorem v^θ (which we normalize by $v^\theta(x_0) = 0$ for a point x_0 with $|x_0| = 2$, and $\nabla v^\theta \to 0$ as $x \to \infty$) and hence u^θ depends differentiably on θ.

The claimed bound for u^θ is an immediate consequence of elliptic regularity estimates. Partial holomorphy however requires an argument. In the next step we connect derivatives with respect to θ to derivatives with respect to x. This connection is obvious if the solution u is partially holomorphic: Recall that

$$u^\theta(x) = u(x + i\theta\phi(x))$$

Let $1 \le L \le n_1$ and

$$\dot u := \partial_{\theta_L} u^\theta = i\phi\partial_{z_L} u = i\phi\partial_{X_L} u^\theta$$

where the second and third identity assume holomorphy and

$$\widetilde u := i\phi\partial_{X_L} u^\theta.$$

We suppress θ in the notation of $\widetilde u$ and $\dot u$. The Cauchy Riemann equations are equivalent to

$$\dot u = \widetilde u, \tag{37}$$

which we have to verify without assuming holomorphy. An easy calculation shows that (see (33) for a definition of the vector fields)

$$\partial_{\theta_L}\partial_{X_j}\psi(x) = -\frac{i\phi_j}{1+i\theta\nabla\phi}\partial_{X_L}\psi(x)$$

hence differentiating the differential equation gives

$$L^\theta \dot u = \partial_{X_k}\Big(a^{km}(\frac{i\partial_m\phi}{1+i\theta\nabla\phi})\partial_{X_L}u\Big) + (\frac{i\partial_k\phi}{1+i\theta\nabla\phi})\partial_{X_L}\Big(a^{kl}\partial_{X_l}u\Big) + \partial_{\theta_L}f^\theta \tag{38}$$

One easily checks that

$$\partial_{X_j}\phi = \frac{\partial_{x_j}\phi}{1+i\theta\nabla\phi},$$

and

$$\big[\partial_{X_j},\partial_{X_l}\big] = 0.$$

As a consequence many of the expressions commute when we apply L^θ to $\widetilde{u}$:

$$\begin{aligned} L^\theta\widetilde{u} =& \partial_{X_j}a^{jk}\partial_{X_k}(i\phi\partial_{X_L}u) \\ =& \partial_{X_j}a^{jk}\frac{i\partial_{x_k}\phi}{1+i\theta\nabla\phi}\partial_L u + \frac{i\partial_{x_j}\phi}{1+i\theta\nabla\phi}\partial_{X_L}a^{jk}\partial_{X_k}u + i\phi\partial_{\theta_L}f^\theta. \end{aligned} \tag{39}$$

Solutions to equation (38) with compact support are unique. Since $\dot u - \widetilde{u}$ is a solution with compact support to the homogeneous equation and since $\partial_{\theta_L}f^\theta = i\phi\partial_{X_L}f^\theta$ for holomorphic functions f we obtain $\dot u = \widetilde{u}$. □

There are only minor changes in the scalar case with $n_2 = 1$ and for coefficients depending measurably on x_n. Consider

$$\partial_i a^{ij}(x_n)\partial_j u = \partial_j F^j + f \tag{40}$$

where the coefficients a^{ij} are uniformly elliptic and measurable.

Proposition 4.3. *Proposition 4.2 holds for* (40) *if we replace the function spaces by $C^s_{*,\delta}$ (with the obvious definition).*

5. Partial analyticity of solutions to the nonlinear equation

There are two steps: First we prove that the Lipschitz solutions have Hölder continuous derivatives. In a second step we characterize the solution as the fixed point of a fixed point problem in a complexioned set, where we use at each step of the iteration the results of the previous section.

5.1. Hölder regularity of derivatives in Theorem 1.

Let u be a Lipschitz continuous weak solution to the elliptic problem

$$\partial_i F^i_k(x,u,Du) = f_k(x,u,Du) \qquad \text{in } B_2(0) \tag{41}$$

under the assumptions of Theorem 1.

Proposition 5.1. *Under the assumptions of Theorem 1 there exists $s>0$ so that $u \in C^{1,s}(B_{3/2}(0))$.*

Proof. The argument could be iterated to yield (partial) smoothness. This we do not pursue, but we will prove partial analyticity in this section. We rewrite the differential equation in terms $v = u - \bar{u} - b \cdot x$. It satisfies an equation of the same type, but with

$$\|v\|_{C^{0,1}} < \varepsilon.$$

Adding a constant if necessary we assume without loss of generality

$$F^i(0,0,0) = 0.$$

We rewrite the equation (41) as

$$\partial_i a^{ij}_{kl} \partial_j v^l = \partial_i G^i_k(x, v, Dv) + f_k(x, v, Dv)$$

where by the assumptions of Theorem 1

$$\begin{aligned} &\|f_k(x, u(x), Du(x))\|_{L^\infty} + \|D_{u,p} f_k(x, u(x), Du(x))\|_{L^\infty} \le \rho \\ &G^i(0,0,0) = 0 \qquad |G^i_k(x, u, p_2) - G^i_k(x, u, p_1)| \le \varepsilon |p_2 - p_1|. \end{aligned}$$

Let u_1 be the solution to

$$\partial_i a^{ij}_{kl} \partial_j u^l_1 = f_k(x, u, Du).$$

For $\sigma > 0$ there exists a solution which satisfies

$$\|Du_1\|_{\dot{C}^\sigma} \le c\rho$$

Then $u_2 = u - u_1$ satisfies

$$\partial_i a^{ij} \partial_j u^l_2 = \partial_i G^i_k(x, u_1 + u_2, Du_1 + Du_2).$$

Let $h \in \mathbb{R}^n$ be small and $v_h = u_2(x+h) - u_2(x)$. Then

$$\partial_i a^{ij} \partial_j v_h = \partial_i \left[\int_0^1 \frac{\partial G^i_k}{\partial P^j_l}(x, u, Du_1 + Du_2 + \lambda(Dv^h)) d\lambda \right] \partial_j v^l_h + H^i \Bigg]$$

with

$$\begin{aligned} H^i = &G^i_k(x+h, u(x+h), Du_1(x+h) + Du_2(x+h)) \\ &- G^i_k(x, u(x), Du_1(x) + Du_2(x+h)). \end{aligned}$$

Under the assumptions of the theorem

$$\|H\|_{C^s} \le c\rho.$$

Let η be a cutoff function and $w = \eta v_h$. Then by the Calderón-Zygmund estimate

$$\|Dw\|_{L^p} \le c\rho |h|^s + \varepsilon \|Dw\|_{L^p}$$

and hence

$$\|Dw\|_{L^p} \le c(\rho + \varepsilon)|h|^s.$$

By Morrey's estimate

$$\|w\|_{C^{1-\frac{n}{p}}} \le c(\rho + \varepsilon)|h|^s$$

and hence

$$|u_2(x+h) - 2u_2(x) + u_2(x-h)| \le c(\rho + \varepsilon)(|h|^{1-\frac{n}{p}} + h^\sigma))$$

with an exponent s if p is sufficiently large, h is small and x is in the interior. This bound implies

$$\|Du_2\|_{C^{s-\frac{n}{p}}(B_{3/2}(0))} \le c(\rho + \varepsilon).$$

This completes the proof. $\square$

Let u be a Lipschitz continuous weak solution to the elliptic problem

$$\partial_i F^i_k(x,u,Du) = f_k(x,u,Du) \tag{42}$$

under the assumptions of Theorem 2. This requires only minor modifications and we state the result.

Proposition 5.2. *Under the assumptions of Theorem 2 there exists $s>0$ so that $Du \in C^s_*(B_{3/2})$.*

5.2. **The nonlinear equation: Analyticity.** In this subsection we prove Theorem 1 and Theorem 2. The arguments are again very similar.

Proof of Theorem 1. We work under the assumptions of Theorem 1, specifying several small parameters along the way. By Proposition 5.1 the weak solution has Hölder continuous derivatives in $B_{3/2}(0)$. We subtract $bx + v(0)$ to reduce the problem to the special situation $b=0$ and $v(0)=0$. Let $\eta \in C_0^\infty(B_{3/2}(0))$ be identically 1 in $B_1(0)$. Then

$$\widetilde{u} = \eta u$$

satisfies

$$\partial_i a^{ij}\partial_j \widetilde{u} = \partial_i \widetilde{F}^i + \widetilde{f}$$

where

$$\widetilde{F}^i = \eta F^i(x,(1-\eta)u + v, D((1-\eta)u) + D\widetilde{u}) - a^{ij}\partial_j\widetilde{u} + a^{ij}(\partial_j\eta)u$$

and

$$\widetilde{f} = (\partial_i\eta)\Big[F^i(x,u,Du) - a^{ij}\partial_j u\Big] + \eta f(x,(1-\eta)u+\widetilde{u}, D(1-\eta)u + D\widetilde{u}).$$

We characterize $\widetilde{u}$ as a fixed point of the map which maps v to the solution to

$$\partial_i a^{ij}\partial_j \widetilde{u} = \partial_i \widetilde{F}^i(x,v,Dv) + \widetilde{f}(x,v,Dv) \tag{43}$$

where we suppress the dependence on u, which is trivial in $B_1(0)$. We normalize $\widetilde{u}$ by choosing a point x_0 outside the ball with $\widetilde{u}(x_0)=0$ and require $\nabla\widetilde{u}\to 0$ as $x\to\infty$.

For small δ and ε to be chosen later we define

$$\begin{aligned} X = \Big\{ u \in C^{1,s}_\delta(U) : \sup_{|\theta|\le\delta} \|u^\theta\|_{C^1(\mathbb{R}^n)} < 2\varepsilon,\\ \|u^\theta\|_{C^{1,s}(\mathbb{R}^n)} \le R, u(x_0)=0, \nabla u(x)\to 0 \text{ as } x\to\infty \Big\} \end{aligned}$$

for some R to be chosen later.

As in Proposition 4.2

$$\begin{aligned} \|Dv - D\widetilde{u}^0\|_{C^s_\delta} \le & c \sup_{|\theta|\le\delta} \Big(\|\widetilde{F}^\theta(x,\widetilde{u}^\theta, D\widetilde{u}^\theta) - F(x,\widetilde{u}^0, D\widetilde{u}^0)\|_{C^s} \\ & + \|\widetilde{f}^\theta(x,\widetilde{u}^\theta, D\widetilde{u}^\theta) - f(x,\widetilde{u}^0, D\widetilde{u}^0)\|_{sup}\Big). \end{aligned}$$

By the triangle inequality

$$\begin{aligned} &\|F^\theta(x' + i\theta(1-|x|^2)^3_+, x'', u^\theta, Du^\theta) - F(x,u,Du)\|_{C^s} \\ &\quad \le \|F^\theta(x' + i\theta(1-|x|^2)^3_+, x'', u^\theta, Du^\theta) - F(x,u^\theta,Du^\theta)\|_{C^s} \\ &\qquad + \|F(x,u^\theta,Du^\theta) - F(x,u,Du)\|_{C^s}. \end{aligned}$$

A straight forward estimate gives for $\rho > \varepsilon$

$$\|F^\theta(x' + i\theta(1-|x|^2)^3_+, x'', u^\theta, Du^\theta) - F(x, u^\theta, Du^\theta)\|_{C^s} \le c(\theta + (\rho+\varepsilon)\|Du^\theta\|_{C^s})$$

and

$$\|F(x, u^\theta, Du^\theta) - F(x, u, Du)\|_{C^s} \le (\rho+\varepsilon)\|Du^\theta - Du\|_{C^s}$$

which implies

$$\|Du\|_{C^s_*} \le c\|u^0\|_{C^{1,s}} + c(\rho+\varepsilon)R$$

We choose $R = 2\|u^0\|_{C^{1,s}(B_{3/2})}$. The similar L^2 estimate gives

$$\|D(u-u^0)\|_{L^2} \le c(\theta + (\rho+\varepsilon)\|D(u-u^0)\|_{L^2}$$

and hence

$$\|D(u-u_0)\|_{L^2} \le c|\theta|$$

Lemma 5.3. *Let $|B_1|$ be the volume of the unit ball. Then*

$$\|f\|_{sup} \le 2|B_1|^{-\frac{s}{2s+n}}\|f\|_{L^2}^{\frac{s}{s+n/2}}\|f\|_{C^{\frac{n/2}{s+n/2}}}.$$

Proof. Clearly

$$|f(x)| \le r^s\|f\|_{\dot C^s} + |B_1|^{-1}r^{-n}\int_{B_r(X)}|f|dx.$$

By Hölder's inequality

$$|f(x)| \le r^s\|f\|_{\dot C^s} + |B_1|^{-1/2}r^{-n/2}\|f\|_{L^2}.$$

We optimize with respect to r and arrive at the assertion of the lemma. □

Thus

$$\|D(u-u^0)\|_{sup} \le c|\theta|^{\frac{s}{s+n/2}}\|u^0\|_{C^{1,s}}^{\frac{n/2}{s+n/2}}.$$

Choosing θ sufficiently small implies

$$\|D(u-u^0)\|_{sup} \le \varepsilon.$$

Then the fixed point map maps a ball in X to itself. It is easy to see that it is a contraction in the norm

$$\sup_\theta \|u(x' + i\theta(1-|x|^2)^3_+, x'')\|_{H^1}.$$

□

Proof of Theorem 2. There are only minor changes for Theorem 2. We define

$$\bar v(x) = u(x_n) + \sum_{i=1}^{n-1} b^i x_i$$

and define

$$v = u - \bar v$$

Then

$$\|v\|_{C^{0,1}(U)} < \varepsilon$$

at least after choosing an appropriate possibly smaller set U. By Proposition 5.2 for any multiindex in $\mathbb{R}^{n-1}$

$$\|\partial^{\alpha'}Dv\|_{sup} \le c_{\alpha'}\varepsilon. \tag{44}$$

Moreover v is a weak solution to

$$\partial_i a^{ij}(x_n)\partial_j u = \partial_i G^i(x, u, Du) + \partial_n H(x, u, Du) \tag{45}$$

with H as in the previous section. □

6. The change of coordinates

6.1. **Theorem 1 implies Theorem 3.** Let $u \in C^1(U, \mathbb{R}^N)$ satisfy the assumptions of Theorem 3. In particular we assume that $(\partial_{x_{n_1+i}} u^k)_{1\le i,k\le N}$ is invertible. We define a diffeomorphism

$$\Xi : \mathbb{R}^n \ni x \to y = (x', u(x)) \in \mathbb{R}^n.$$

The Jacobi matrix is

$$D\Xi(x) = \left(\frac{\partial y_i}{\partial x_j}\right)_{1\le i,j\le n} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u^1_{x_1} & u^1_{x_2} & \dots & u^1_{x_n} \\ \vdots & \vdots & \ddots & \vdots \\ u^N_{x_1} & u^N_{x_2} & \dots & u^N_{x_n} \end{pmatrix} = \begin{pmatrix} 1_{n_1,n_1} & 0_{n_1 N} \\ D'_x u & D''_x u \end{pmatrix}.$$

We define $v(y) = x''$ and $\Psi : y \to (y', v(y))$. Then

$$\psi \circ \Xi = 1$$

and hence

$$\sum_{l=1}^N (\partial_{y_{n_1+j}} v^l)(\partial_{x_{n_1+l}} u^k) = \delta^k_j$$

and, if $1 \le k \le n_1$

$$\sum_{j=1}^N (\partial_{y_{n_1+j}} v^l)\partial_{x_k} u^j = -\partial_{y_k} v^l$$

It is useful to write these formulas more compact as

$$D''_x u D''_y v = 1 \qquad D''_y v D'_x u = -D'_y v.$$

Then

$$\int F^i_k(x', x'', u, D'_x u, D''_x u)\partial_i \phi_k dx = \int G^l_k(y, v, Dv)\partial_{y_l}\phi_k dy$$

with $p^j_l = \partial_{y_l} v^j$ and

$$G_k(y, v, p)^i = \begin{pmatrix} 1 & 0 \\ (p'')^{-1}p' & (p'')^{-1} \end{pmatrix}_{il} F^l_k(y', v, y'', -(p'')^{-1}p', (p'')^{-1}) \det p''.$$

Lemma 6.1. *The weak system*

$$\partial_i G^i_k(x, v, Dv) = g(x, v, Dv)$$

is elliptic.

Proof. We have to verify that

$$\left| \left(\frac{G^i_k}{\partial p^j_l}\xi_i\xi_l\eta_k\right)_j \right| \ge \kappa|\xi|^2|\eta|. \tag{46}$$

Let $A = (a_{ij})_{1\le ij\le m}$ be a square matrix. Then, from the expansion of the determinant, if $1 \le i, j, l \le n$

$$\sum_l \left(\frac{\partial \det A}{\partial a_{il}}\right)_{il} a_{lj} = \sum_i \left(\frac{\partial \det A}{\partial a_{li}}\right)_{il} a_{lj} = \det A\delta_{ij}$$

and thus

$$\det D''v D\Xi = \begin{pmatrix} \det D''v\, 1_{n_1\times n_1} & 0_{n_1\times n_2} \\ -\sum_{l=1}^{N}\left(\frac{\partial \det D''v}{\partial(\partial_{y_i} v^l)}\partial_{y_j} v^l\right)_{n_1<i\le N, 1\le j\le n_1} & \frac{\partial \det D''v}{\partial(\partial_{y_j} v^{l-n_1})} \end{pmatrix}_{jm}$$

The identity (for $1 \le m,k,l \le N$)

$$\frac{\partial^2 \det D''v}{\partial(\partial_{y_{n_1+m}} v^l)\partial(\partial_{y_{n_1+l}} v^k)} + \frac{\partial^2 \det D''v}{\partial(\partial_{y_{n_1+k}} v^l)\partial(\partial_{y_{n_1+l}} v^m)} = 0$$

is a consequence of the expansion of the determinant. We claim that for $1 \le k \le N, 1 \le m \le n$

$$(47)\qquad \begin{aligned} 0 =& \frac{\partial}{\partial p_i^k} \begin{pmatrix} \det p'' 1 & 0 \\ -\left(\sum_{l=n_1+1}^{n} \frac{\partial \det p''}{\partial p_m^l} p_j^l\right)_{n_1<m\le n, 1\le j\le n_1} & \left(\frac{\partial \det p''}{\partial p_m^j}\right)_{n_1<j,m\le n} \end{pmatrix}_{mj} \\ &+ \frac{\partial}{\partial p_j^k} \begin{pmatrix} \det p'' 1 & 0 \\ -\left(\sum_{l=n_1+1}^{n} \frac{\partial \det p''}{\partial p_m^l} p_i^l\right)_{n_1<m\le n, 1\le i\le n_1} & \left(\frac{\partial \det p''}{\partial p_j^m}\right)_{n_1<j,m\le n} \end{pmatrix}_{mi} \end{aligned}$$

As a consequence of the considerations above the sum in (47) vanishes for $n_1 < i,j \le n$. The claim is trivial for $1 \le i,j \le n_1$ since only the block matrix on the lower left corner depends linearly on those coefficients. It remains to consider $1 \le i \le n_1 < j \le n$. The claim (47) follows now from

$$\begin{aligned} \frac{\partial(d\Xi_i^m)}{\partial(\partial_{y_j} v^k)} + \frac{\partial(d\Xi_j^m)}{\partial(\partial_{y_i} v^k)} =& \frac{\partial \det D''v}{\partial(\partial_{y_j} v^k)}\delta_{im} - \frac{\partial}{\partial_{(\partial_{y_i} v^k)}} \sum_{l=1}^{n_1} \frac{\partial \det D''v}{\partial_{y_{y_j}} v^l}\partial_{y_m} v^l \\ =& \frac{\partial \det D''v}{\partial(\partial_{y_j} v^k)}\delta_{im} - \frac{\partial \det D''v}{\partial(\partial_{y_j} v^k)}\delta_{im} = 0. \end{aligned}$$

Then

$$\begin{aligned} b_{kl}^{ij}\xi_i\xi_j =& (\det D''v)^{-1}(D\Xi^T\xi)_{i'}\xi_{j'} \frac{\partial F_k^{i'}(x,v,-(D''v)^{-1}D'v,(D''v)^{-1})}{\partial(\partial_{y_j} v^l)} \\ =& -(D\Xi^T\xi)_{i'}(D\Xi^T\xi)_{j'} a_{kl'}^{i'j'} \frac{\partial \det D''v}{\partial(\partial y_{l'} v^l)} \end{aligned}$$

and (46) is an immediate consequence. □

To complete the proof we fix a point x_0 and define $u^r(x) = r^{-1}u((x-x_0)/r)$. The smallness assumptions are satisfied if we choose r sufficiently small.

6.2. **Theorem 2 implies Theorem 4.** Specializing the previous calculation we obtain the transformed problem

$$(48)\qquad \partial_i G^i(y,v,Dv) = \widetilde{g}(y,v,Dv)$$

where with $y = (y', y'')$

$$G^i(y,v,Dv) = v_n F^i(y', v, y'', -(D_t v)^{-1} D_{y'} v, (D_t v)^{-1})$$

if $1 \le i < n$ and

$$\begin{aligned} G^n(y,v,Dv) = &- v_i F^i(y', v, y'', -(D_t v)^{-1} D_{y'} v, (D_t v)^{-1}) \\ &+ F^n((y', v, y'', -(D_t v)^{-1} D_{y'} v, (D_t v)^{-1})), \end{aligned}$$

$$g(y,v,Dv) = v_n f(y', v, y'', -(D_t v)^{-1} D_{y'} v, (D_t v)^{-1}).$$

By the previous section the equation is elliptic. The notion of ellipticity in this scalar context simplifies to

$$a^{ij}\xi_i\xi_j \ge \kappa|\xi|^2$$

with

$$a^{ij}(y,v,Dv) = \frac{\partial G^i(y,v,p)}{\partial p_j}.$$

Now we change the notation, replace $n-1$ by n and denote y' by x and $x_n = t$, use the index 0 for the time component and denote G again by F and g by f so that the transformed equation becomes

$$\partial_i F^i(t,x,v,Dv) = f(t,x,v,Dv)$$

where F and f depend analytically on x, v and Dv uniform in t. Hence they admit an extension into a partial complexification of the domain.

In order to apply Theorem 2 we have to ensure that

$$v - \sum_{i=1}^n b_i x_i - v_0(t)$$

has a small Lipschitz constant for some constants b and a Lipschitz function v_0.

Lemma 6.2. *There exists $s > 0$ so that*

$$D_x v \in C^s, \qquad \partial_t v \in C^s_*$$

Proof. For $1 \le k \le n$ one obtains (to do it rigorously one has to consider finite differences)

$$\partial_i a^{ij}(t,x,v,Dv)\partial_j(\partial_k v) = -\partial_i(\partial_u F \partial_k v + \partial_{x_k} F) + \partial_{x_k} f$$

and, by the Hölder regularity result of De Giorgi, Nash and Moser

$$\partial_k u \in C^s$$

for some $s > 0$. Now we apply Caccioppoli's inequality in balls $B_r(t,x)$ to get

$$\|D_{t,x}\partial_k u\|_{L^2(B_r)} \le c r^{\frac{n}{2}+s-1}$$

and hence

$$\|\partial_k F^i\|_{L^2(B_r)} \le c r^{\frac{n}{2}+s-1}.$$

This in turn gives

$$\|\partial_t F^0\|_{L^2(B_r)} \le c r^{\frac{n}{2}+s-1}$$

and hence together

$$\|F^0 - F^0_{B_r(x)}\|_{L^2} \le c r^{\frac{n}{2}+s}.$$

In particular

$$|F^0_{B_{2^{-k}}(x)} - F^0_{B_{2^{1-k}}(x)}| \le c 2^{-ks}$$

and

$$F^0 \in C^s$$

By assumption we can solve

$$F^0(t, x, u, Du) = f$$

for $\partial_t u$ and get

$$\partial_t u = \phi(t, x, u, D'u, f)$$

where ϕ is analytic in all variables besides t. Thus

$$\partial_t u \in C^s_*.$$

□

Decreasing the domain if necessary we can ensure the assumption of Theorem 2. Thus v is analytic with respect to x. The level surfaces at level u_0 are parametrized by

$$y' \to (y', v(y, u_0)).$$

Moreover the derivatives of v are holomorphic with respect to x. The claim on analyticity of the derivatives holds since

$$u_{x_n} = (v_t)^{-1}$$

and, for $j < n$

$$u_{x_j} = -v_{y_n}^{-1} v_{y_j}.$$

6.3. **Water waves with surface tension.**

Proof of Theorem 2.4, completion. If $\kappa \neq 0$ we observe that at the free boundary

$$\partial_{y_1} \frac{u_{y_1}}{\sqrt{1 + u_{y_1}^2}}$$

is bounded, hence $u_{y_2=0} \in C^{1,1}$. With a small modification of the proof of Theorem 2 we obtain $Du \in C^s_*$: we need an additional estimate for a linear problem in Lemma 6.3 below. □

Lemma 6.3. *Let $(a^{ij}(x_n))_{1 \le i,j \le n}$ and $(b^{ij})_{1 \le i,j < n}$ be bounded uniformly positive definite matrices. We consider the system*

$$\sum_{i,j=1}^{n} \partial_i a^{ij} \partial_j u = \sum_{i=1}^{n} \partial_i f^i$$

in $x_n > 0$ with the boundary condition

$$\sum_{i,j=1}^{n-1} \partial_i (b^{ij} \partial_j u) = \sum_{i=1}^{n-1} \partial_i g^i \qquad \text{on } \{x_n = 0\}.$$

Suppose that g has a holomorphic extension and that f has a partially holomorphic extension. Then the same is true for u and

$$\|Du\|_{C^s_{*,\delta}} \le c\Big(1 + \|f\|_{C^s_{*,\delta}} + \|g\|_{C^s_\delta}\Big)$$

provided δ is sufficiently small.

A similar boundary value problem has been considered in [15].

Proof. We first obtain the interior bound at the boundary

$$\|Du|_{x_n=0}\|_{C^s_\delta} \le c(1 + \|g\|_{C^s_\delta}$$

and then by the boundary analogue

$$\|Du\|_{C^s_\delta} \le c\left(\|f\|_{C^s_\delta} + \|g\|_{s,\delta}\right),$$

compare the proof of the analogous statements Proposition 4.2 and 4.3. □

There is no change in the setting of Theorem 4 resp. for the water wave problem with surface tension.

References

[1] S. Agmon, A. Douglis, and L. Nirenberg. "Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II". *Comm. Pure Appl. Math.* 17 (1964), pp. 35–92.

[2] Ioannis Athanasopoulos and Luis A. Caffarelli. "A theorem of real analysis and its application to free boundary problems". *Comm. Pure Appl. Math.* 38.5 (1985), pp. 499–502.

[3] S. Bernstein. "Demonstration du théorème de M. Hilbert sur la nature analytique des solutions des équations du type elliptique sans l'emploi des séries normales". *Math. Z.* 28.1 (1928), pp. 330–348.

[4] S. Bianchini and N.A. Gusev. "Steady nearly incompressible vector fields in 2D: chain rule and renormalization". *arXiv preprint arXiv:1408.2932* (2014).

[5] Xavier Cabré and Sagun Chanillo. "Stable solutions of semilinear elliptic problems in convex domains". *Selecta Math. (N.S.)* 4.1 (1998), pp. 1–10.

[6] Thierry Colin. "Rigorous derivation of the nonlinear Schrödinger equation and Davey-Stewartson systems from quadratic hyperbolic systems". *Asymptot. Anal.* 31.1 (2002), pp. 69–91.

[7] Adrian Constantin and Joachim Escher. "Analyticity of periodic traveling free surface water waves with vorticity". *Ann. of Math. (2)* 173.1 (2011), pp. 559–568.

[8] Peter Constantin, Vlad Vicol, and Jiahong Wu. "Analyticity of Lagrangian trajectories for well posed inviscid incompressible fluid models". *arXiv preprint arXiv:1403.5749* (2014).

[9] Jean-Marc Delort. "Existence de nappes de tourbillon en dimension deux". *J. Amer. Math. Soc.* 4.3 (1991), pp. 553–586.

[10] L. C. Evans and S. Müller. "Hardy spaces and the two-dimensional Euler equations with nonnegative vorticity". *J. Amer. Math. Soc.* 7.1 (1994), pp. 199–219.

[11] H. Gourgeon and J. Mossino. "Sur un problème à frontière libre de la physique des plasmas". *Ann. Inst. Fourier (Grenoble)* 29.4 (1979), pp. ix, 127–141.

[12] Michael Grüter and Kjell-Ove Widman. "The Green function for uniformly elliptic equations". *Manuscripta Math.* 37.3 (1982), pp. 303–342.

[13] D. Kinderlehrer and L. Nirenberg. "Regularity in free boundary problems". *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 4.2 (1977), pp. 373–391.

[14] Herbert Koch. "Transport and instability for perfect fluids". *Math. Ann.* 323.3 (2002), pp. 491–523.

[15] Herbert Koch, Giovanni Leoni, and Massimiliano Morini. "On optimal regularity of free boundary problems and a conjecture of De Giorgi". *Comm. Pure Appl. Math.* 58.8 (2005), pp. 1051–1076.

[16] S. Kowalevsky. "Zur Theorie der partiellen Differentialgleichungen". *Dissertation, Göttingen, appeared in Berlin, Georg Reimer* (1874).

[17] Olga A. Ladyzhenskaya and Nina N. Ural′tseva. *Linear and quasilinear elliptic equations*. Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis. Academic Press, New York-London, 1968, pp. xviii+495.

[18] Hans Lewy. "Neuer Beweis des analytischen Charakters der Lösungen elliptischer Differentialgleichungen". *Math. Ann.* 101.1 (1929), pp. 609–619.

[19] Leon Lichtenstein. "Über einige Hilfssätze der Potentialtheorie. I". *Math. Z.* 23.1 (1925), pp. 72–88.

[20] Pierre-Louis Lions. *Mathematical topics in fluid mechanics. Vol. 2.* Vol. 10. Oxford Lecture Series in Mathematics and its Applications. Compressible models, Oxford Science Publications. New York: The Clarendon Press Oxford University Press, 1998, pp. xiv+348.

[21] Andrew J. Majda and Andrea L. Bertozzi. *Vorticity and incompressible flow*. Vol. 27. Cambridge Texts in Applied Mathematics. Cambridge: Cambridge University Press, 2002, pp. xii+545.

[22] Charles B. Morrey Jr. *Multiple integrals in the calculus of variations*. Classics in Mathematics. Reprint of the 1966 edition [MR0202511]. Berlin: Springer-Verlag, 2008, pp. x+506.

[23] I. G. Petrowsky. "Sur l'analyticité des solutions des systèmes d'équations différentielles". *Rec. Math. N. S. [Mat. Sbornik]* 5(47) (1939), pp. 3–70.

[24] Philippe Serfati. "Structures holomorphes à faible régularité spatiale en mécanique des fluides". *J. Math. Pures Appl. (9)* 74.2 (1995), pp. 95–104.

[25] H. Shahgholian. "Regularity issues for semilinar PDE's (a narrative approach) dedcated to Nina Nikolaevna Uraltseva". *preprint* (2014).

[26] Vladislav Zheligovsky and Uriel Frisch. "Time-analyticity of Lagrangian particle trajectories in ideal fluid flow". *J. Fluid Mech.* 749 (2014), pp. 404–430.

Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

E-mail address: koch@math.uni-bonn.de

Aix-Marseille Université,, I2M, 39 rue F. Joliot-Curie, 13453 Marseille, France,
E-mail address: nikolay.nadirashvili@univ-amu.fr